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CERN EUROPEAN ORGANIZATION FOR NUCLEAR RESEARCH

MODERN PHYSICS FROM AN ELEMENTARY POINT OF VIEW

V.F. Weisskopf

Lectures given in the
Summer Vacation Programme
1969

GENEVA

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*) Notes taken by J.P. Lagnaux, E. Athanasoula, B. Bramsen and E. Mandelkow.

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INTRODUCTION

One of the greatest advances in physics in the last century was to give answers to certain questions which had often been asked before, such as "How big are atoms?", "What are the properties of ordinary matter, such as compressibility and surface tension?". All of these quantities had previously been measured, but now we know why they have their observed values. We can say the same, with somewhat less assurance, about nuclear structure and about astronomical questions such as the stars (the life, the size, and the temperature of stars), and to an even lesser extent, or practically not at all, about the branch of physics we are dealing with here at CERN, namely the question of the elementary particles. This is why this account will contain very little about elementary particles; simply because from the point of view of understanding what is really going on, we unfortunately know very little. But there is a purpose in what we are about to do, because in looking at the structure of the rest of physics we build up an idea of what to expect for the physics of elementary particles, and we can see just how far we are from this ideal situation. In fact, as far as explaining the observable properties of elementary particles, the situation today is about the same as it was in the case of the atoms before the turn of the century.

Physics -- atomic physics, nuclear physics, molecular physics, solid-state physics -- is a very complicated thing. The essential idea of this account is to go back to the main fundamental ideas which I feel have somehow been lost in the wealth of complicated theory and mathematics. Therefore I shall be very qualitative. We shall only calculate order of magnitude answers to all the questions that we shall ask. Whenever there is an equal sign, it will not mean "equal", only "the same order of magnitude as". Factors such as 2, π, etc., will be neglected. What a wonderful life, but only from the philosophical point of view. In the daily work of the physicist, constants are, of course, important. However, I would like you to consider this account as a form of "higher entertainment".

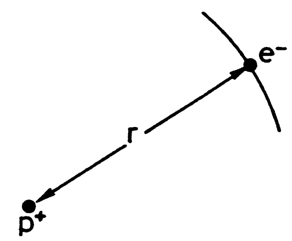
We shall start with a very simple question: "How big is an atom?". In the nineteenth century, people knew quite well that atoms existed. They even knew how to count them, and knew roughly their size (from the number in a known volume of a solid or a liquid where they are tightly packed). This size was, however, only a number and the most fundamental progress of quantum theory was that it revealed the size of the atoms in terms of fundamental magnitudes, such as the electronic charge e, the mass of the electron m_e and the quantum of action (Planck's constant) $\hbar$.

Let us calculate, in a very simple way which many of you will know, the size and the energy of the hydrogen atom, and look at the essential physical principles and constants involved.

1. ATOMIC AND MOLECULAR PHYSICS

1.1 The hydrogen atom

We shall now derive the size and the energy of the H-atom. This consists of two particles, a proton and an electron, bound together by an attractive Coulomb force.



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The electron, circling around the proton, has a potential energy $V = -e^2/r$ and a kinetic energy $K = p^2/2m_e$. Here r is the average distance from the proton, and p is the momentum of the electron. But we know from quantum mechanics that $p = \hbar/\lambda\!\!\!{}^-$, where $\lambda\!\!\!{}^-$ is the De Broglie wavelength of the electron. In a bound state this wavelength is not well defined. In a quantum state without nodes, which is of the size r, we can write $\lambda\!\!\!{}^- \sim r$ (because a Schrödinger wave pattern of size r consists mostly, if one makes a Fourier analysis, of waves of wavelength λ which are of the same size as this wave pattern, if it is a simple one). From this we derive that

$$K = \frac{\hbar^2}{2m_e r^2} \ . \tag{1.1}$$

Therefore the total energy is

$$E = -\frac{e^2}{r} + \frac{\hbar^2}{2m_e r^2}$$

or

$$E = -\frac{A_0}{r} + \frac{B_0}{2r^2} \ ,$$

$$\text{with} \quad A_0 = e^2 \quad \text{and} \quad B_0 = \hbar^2/m_e .$$

The electron tries to keep a compromise between the electrostatic attraction (which pulls it towards the proton) and the quantum kinetic energy [Eq. (1.1)], which would be low if r is large. The compromise is reached when the total energy is kept to a minimum. The corresponding radius is found by setting

$$\left.\frac{dE}{dr}\right|_{a_0} = 0 \ ,$$

which leads to

$$a_0 = \frac{B_0}{A_0} = \frac{\hbar^2}{m_e e^2} = 0.53 \text{ Å} .$$

One gets thus

$$E_0 = -\frac{A_0^2}{2B_0} = -\frac{m_e e^4}{2\hbar^2} \simeq -13.6 \text{ eV} = -1 \text{ Ry} \ . \tag{1.2}$$

We call a_0 the Bohr radius and Ry the Rydberg.

Now let us look at the first excited state, i.e. that wave function which has a node. It is clear that the wavelength will not be equal to r, as it was before, but to $r/2$, so that the kinetic energy K and the constant B will be four times bigger:

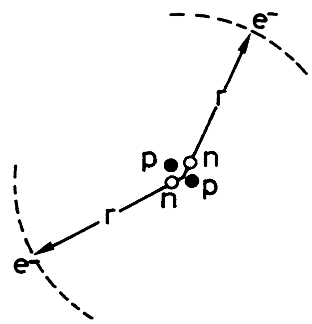
$$K_1 = 4\ K_0$$
$$B_1 = 4\ B_0 \ .$$

This means that the binding energy will be four times smaller and the radius four times larger:

$$E_1 = \frac{1}{4} E_0 = -0.25 \text{ Ry} \ ; \qquad r_1 = 4a_0 \ .$$

This is then the origin of the Balmer spectrum of the hydrogen atom. If an orbit is n wavelengths long, the energy will be $(1/n^2)E_0$ and the radius n^2a_0, so that the energy released by a transition from the excited state to the ground state is $(1 - 1/n^2)E_0$.

1.2 The helium atom



Consider two electrons in the ground state circling around the nucleus. Their combined potential energy in the field of the nucleus is

$$V = -\frac{(2e)^2}{r} = -4e^2/r$$

and their kinetic energy is

$$T = 2\,\frac{\hbar^2}{2m_e r^2} ,$$

so that we have

$$E = -4\,\frac{e^2}{r} + 2\,\frac{\hbar^2}{2m_e r^2} .$$

From this we can derive as before the minimum energy and radius:

$$E = \frac{4^2}{2}\,E_0 = -8\text{ Ry} ; \qquad r = \frac{2}{4}\,a_0 = 0.5\,a_0 .$$

However, this is not quite true, because we did not take into account the repulsion between the electrons. This lowers the binding energy. We assume that on the average the electrons are at a distance r_{eff} apart, with r_{eff} somewhere between r and the full diameter 2r. Let us work with a ratio $r_{eff} : r = 10 : 6$, or $r_{eff} = r/0.6$. Then the repulsive potential is

$$+\frac{e^2}{r_{eff}} = +0.6\,\frac{e^2}{r}$$

and the total energy is

$$E = \frac{e^2}{r}\,(-4 + 0.6) + 2\,\frac{\hbar^2}{2m_e r^2} .$$

Now A must be replaced by 3.4 A and we get

$$E = \frac{(3.4)^2}{2}\,E_0 = -5.8\text{ Ry} ; \qquad r = \frac{2}{3.4}\,a_0 \simeq 0.6\,a_0 .$$

This energy is almost exactly the binding energy of the two electrons in helium, thus justifying our assumption for r_{eff}.

1.3 The neon atom

Here we have two K-shell (n = 1) and eight L-shell (n = 2) electrons surrounding a nucleus of charge Z = 10.

The K-orbit radius is very small because it goes as 1/A and A is proportional to the charge -- here 10 -- so that it will be roughly 1/10 to the hydrogen radius. So we are going to assume an effective Z, $Z_{eff} = 8$ and only eight electrons, all in the L shell, which has states with one node.

The attractive potential of all L-electrons in the field of the nucleus is $-(8e)^2/r$. To calculate the repulsive potential, we shall consider, as in the helium atom, that $r_{eff} = r/0.6$ per pair. We now count the total number of pairs:

$$\frac{N(N-1)}{2} = \frac{8 \times 7}{2} = \frac{56}{2} = 28$$

and get a total energy (for all eight electrons):

$$E = \frac{e^2}{2r}\left[-(8)^2 + 28 \times 0.6\right] + 8 \times 4 \frac{\hbar^2}{2m_e r^2} .$$

Therefore:

$$A = 47\ A_0 \ , \qquad B = 32\ B_0$$

and, as in Eq. (1.2),

$$E = \frac{(47)^2}{8 \times 4} E_0 = -\ 69\ \text{Ry} \ ; \qquad r = \frac{8 \times 4}{47} = \frac{2}{3}\ a_0 \ .$$

In general, for a given Z_{eff}, the total energy and radius of the outer shell is given by

$$E = \frac{\left[Z_{eff}^2 - \frac{Z_{eff}(Z_{eff}-1)}{2} \times 0.6\right]^2}{Z_{eff}\ n^2} = \frac{Z_{eff}\left[Z_{eff} - 0.3(Z_{eff}-1)\right]^2}{n^2}$$

and

$$r = \frac{Z_{eff}\ n^2}{Z_{eff}^2 - \frac{Z_{eff}(Z_{eff}-1)}{2} \times 0.6} = \frac{n^2}{Z_{eff} - (Z_{eff}-1) \times 0.3} .$$

The following table compares our calculations with actual measurements [the energies were taken from Landolt, the radii from Shankland*)].

The general agreement between measured and calculated values justifies our approach. We can now understand the following:

1) The atomic radii increase with n^2 and decrease with Z, so that there is a jump for every new shell (Li, Na, etc.); generally the radii will be of the order of 0.5 to a few Bohr radii;

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*) Landolt-Börnstein Zahlenwerte und Funktionen aus Naturwissenschaften und Technik, Neue Serie, Gesamtherausgabe K.H. Hellwege, Gruppe I, Band 1: Energie-Niveaus der Kerne (Springer Verlag, Berlin, 1961).
R.S. Shankland, Atomic and Nuclear Physics (Macmillan, New York, 1955).

2) The binding energy of one electron in the outer shell increases roughly as Z_{eff}^2 and decreases as n^2, the largest values corresponding to closed shells.

Element	Z	Z_{eff}	n	r[a_0]	E[Ry]	r[a_0]	E[Ry]
				Calculated		Measured	
H	1	1	1	1.0	1.0	1.0	1.0
He	2	2	1	0.6	5.8	0.6	5.8
Li	3	1	2	4.0	0.25	2.8	0.4
Be	4	2	2	2.4	1.4	2.2	2.0
B	5	3	2	1.7	4.3	1.6	5.2
C	6	4	2	1.3	9.6	1.2	10.9
N	7	5	2	1.1	18.0	1.0	19.5
O	8	6	2	0.9	30.5	0.8	31.8
F	9	7	2	0.8	42.0	0.7	48.5
Ne	10	8	2	0.7	69.0	0.6	70.0

1.4 Binding energies of solids

In order to understand some aspects of solid-state physics, which will be discussed later, we want to derive as an example the binding energies and distances of ionic crystals. As an example, we can take Na^+F^-, the lattice of which looks like that shown in Fig. 1:

+ - + - + - + - + -

- + - + - + - + - +

Fig. 1

where at the + places there is a Na^+ and at the - places a F^-. These attract each other by a Coulomb force. In fact, if the ions were points, the lowest energy would be reached if the whole lattice collapsed. But they have radii -- the core radii R -- and the nearest they can come is when the cores touch (because of the Pauli principle, which prevents electrons from taking the same place, or otherwise of having the same quantum numbers, therefore preventing the cores from interpenetrating each other). So there will be a distance between the ions, which we shall call a.

Let us now calculate the binding energy of this +- structure. To do this, one must add up the Coulomb fields of all surrounding ions in a series, with alternative sign terms, which converges. The result is

$$E_c = -0.87\ e^2/a \qquad \text{per ion} .$$

Now if the atoms were exact spheres, then the energy necessary to take them apart at zero temperature would obviously be this Coulomb energy. But of course this is not quite true. If we plot $E_C = f(a)$, we notice that as a gets smaller E_C goes down, and when a reaches the value $r_1 + r_2$ (the sum of the two radii) then the potential goes up rapidly. So what we should use as a binding energy is this minimum and not the Coulomb potential at that point, but if the curve is steep there is no big difference (see Fig. 2):

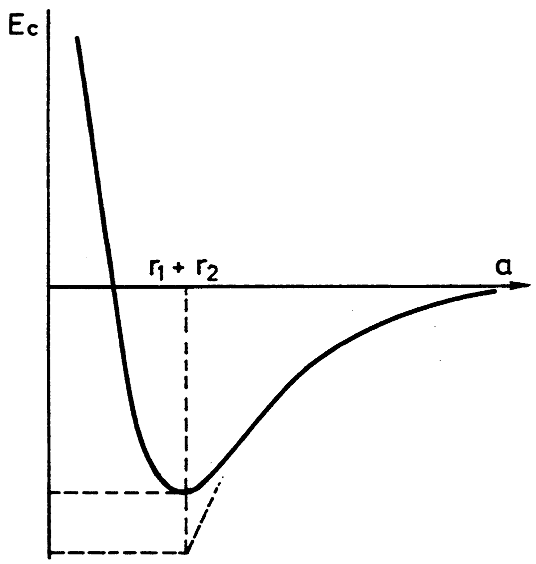


Fig. 2

To get the distance a, one should not add up the two average radii of Na^+ and F^- but the actual ones, which will always be bigger. This can be understood by comparing the radius R and the average radius R_{av} of a sphere of homogeneous charge distribution. By definition

$$R_{av} = \frac{\int_0^R r\rho \, dV}{\int_0^R \rho \, dV} = \frac{2}{3} R .$$

Thus introducing a fudge factor f we have $d = f(r_1 + r_2) \simeq 2f\, a_0$, where $1 < f < 4$. The potential energy per atom in this solid is then:

$$E = -0.87 \frac{e^2}{d} \simeq - \frac{0.87}{2f} \frac{e^2}{a_0} = \eta_s \, Ry$$

and the binding energy $B_S \approx \eta_S$ Ry, with $\eta_S = 0.2, \ldots, 0.8$. We therefore find that the order of magnitude of binding energies in solids is several volts.

This B_S is the energy we have to spend when we want to take an isolated atom of temperature $T = 0$ away from the lattice. Of course if we are at room temperatures ($T \simeq 300°K$) we will need less energy.

In liquids (and by liquids we mean those things that are liquids at room temperature, e.g. water, alcohol), obviously the binding must be less: $\eta_L \approx 0.05\text{-}0.2$.

For those substances that are gases at room temperature, η is very small. For the rare gases He, Ne, A, Kr, Xe, Rn, where there is practically no binding between the atoms because of their closed electron shells, η is almost zero.

1.5 Lattice vibrations

Now let us take a crystal. If it is heated it should vibrate, and the vibration energy per atom should be 3 kT. This follows from the equipartition of energy, which says that each degree of freedom corresponds to an energy of $\frac{1}{2}$ kT. But here we have six degrees of freedom (three positions and three momenta), so 3 kT in all. Let us now consider only one-dimensional displacements. Then we should use kT instead of 3 kT. We would like to find the amplitude of these vibrations.

If the energy $\varepsilon = B$, the particles would separate; this means the amplitude of vibration will be, roughly speaking, equal to

$$b \simeq a/2 \ .$$

Now let us calculate the amplitude of vibration when $\varepsilon < B$. The amplitudes of an oscillator are proportional to the square root of the energy, therefore

$$b = \frac{a}{2}\sqrt{\frac{\varepsilon}{B}} = \frac{a}{2}\sqrt{\frac{kT}{B}} \ .$$

If we use $B \simeq 5$ eV and $T_{room} = 300°$, then $kT = 1/40$ eV and the amplitude is

$$A = \frac{a}{2}\sqrt{\frac{1}{5 \times 40}} \approx \frac{a}{30} \ll a \ .$$

So we see that the vibration amplitudes at room temperature are small compared to the lattice distance.

1.6 Evaporation temperature of solids

One can take a handbook of constants, look up a lot of numbers such as binding energies, elasticity coefficients, melting temperatures, and so on, and then ask oneself: "Why are they that big and not a 100 times smaller or bigger?" This is the question we are now going to ask ourselves regarding evaporation temperatures of solids. These are usually of the order of 3,000 degrees. Let us now calculate them.

We would have been naïve if we had said that to get each atom loose we would have to give it an energy kT equal to its binding energy. If that were so and we used a binding energy of 5 eV, we would get melting temperatures of more than 50,000 degrees, whereas they should be less (1 eV = 10,000°).

Now let us see how one should tackle the problem. The evaporation temperature depends on the pressure of the gas surrounding the solid. We assume that this pressure is one atmosphere. We shall consider a solid lattice of atoms A and above it a gas of the same atoms. We define the evaporation temperature as that temperature at which the number of particles escaping from the surface is equal to the number of particles coming down from the gas. As a rough approximation, we consider the solid as a very dense gas held together within the volume of the solid. The probability that one atom escapes is, according to the Boltzmann statistics, $e^{-B/kT_{ev}}$ times the probability that it comes to the surface. The probability for coming down from the gas to the solid is roughly equal to the probability that a gas atom reaches the surface. Since the density of the gas is 1000 times smaller than the density in the solid (we consider the solid as a gas of high pressure), the chance of reaching the surface is 1000 times smaller from the gas to the solid than from the solid to the gas. Since only a fraction $e^{-B/kT_{ev}}$ of those which reach the surface from the solid will enter the gas, but every one that reaches the surface from the gas will stick to the solid, the number coming down and the number escaping will be equal if

$$e^{-B/kT_{ev}} \sim \frac{1}{1000} ,$$

i.e. $$B/kT_{ev} = \log 1000 = 7$$

or $$kT_{ev} = B/7 .$$

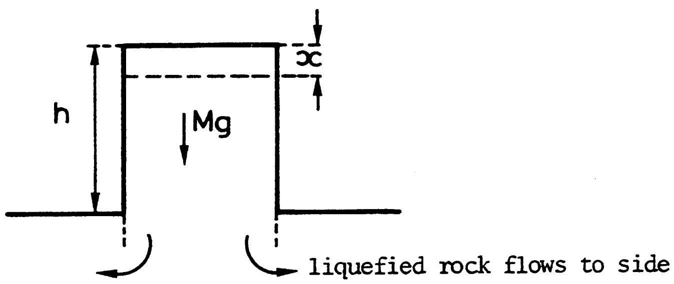
With $B \sim 3$ eV, we get $T \sim 4000°$, which is a little high but of the right order of magnitude.

1.7 Heights of the mountains in terms of fundamental constants

Of course there are high mountains and low mountains. But one may ask, why is the maximum height 10 km (Mount Everest) and not fifty times as much? We see them down to zero height because of erosion and fresh formation. But what is the significance of that order of magnitude of 10 km = 10^6 cm? It is a consequence of the nature of the solid state of rock. It is also connected with the strength of gravity expressed by the gravitational constant, and with the number of nucleons in the Earth, which is 3×10^{51}. Later on we will show that this number is not purely accidental and must lie between certain limits.

Why do the mountains not grow infinitely high? If a mountain gets too high it sinks into the earth beneath it because the material in this earth -- the granite, quartz, or silicon dioxide -- is not strong enough to hold it. The force due to the weight of the mountain is sufficient to break the directionality of the bonds between the atoms in the rock, i.e. it is liquefied and can flow aside so that the mountain sinks. The energy necessary for the liquefaction comes from the potential energy lost by the mountain when it sinks into the ground.

Let the height of the mountain be h and let it sink by a distance x. Let the mass of the mountain be M:



The solid line represents the initial position of the mountain and the dotted line its final position.

Loss in gravitational potential energy = energy needed to liquefy a mass of rock equal to the mass of a height x of the mountain, i.e.

$$Mg\ x = E_{liq} \times nxX \ ,$$

where

M = mass of the mountain

n = number of molecules per unit volume

X = cross-sectional area of the base of the mountain

E_{liq} = liquefaction energy (i.e. latent heat of melting) per molecule.

Cancelling x gives

$$Mg = E_{liq} \times nX \ . \tag{1.3}$$

The right-hand side of this equation has a definite value. Therefore in order to produce liquefaction, i.e. in order for the mountain to sink, M must have a certain minimum value. If M is less than this critical value, the mountain is stable. Therefore the masses of stable mountains are given by

$$Mg \leq E_{liq}\ nX \ . \tag{1.4}$$

Now $M = hX \times n \times m$; m = mass of a molecule of rock = $A\ m_p$, where A is the atomic number of the molecule, i.e.

$$M = hXn\ A\ m_p \ .$$

Substituting for M in Eq. (1.4) gives

$$hXn\ A\ m_p\ g \leq E_{liq}\ nX \ ,$$

i.e.

$$h \leq \frac{E_{liq}}{A\ m_p\ g} \ , \tag{1.5}$$

so that h must be less than the critical value $E_{liq}/A\ m_p\ g$ for the mountain not to sink into the earth.

What is the liquefaction energy E_{liq}? A liquid is, in fact, quite well bonded. When a solid melts, the whole bonds between the atoms are not broken, just the directionality of the bonds. This enables a liquid to flow, whereas a solid cannot because its bonds are held in fixed positions relative to its constituent atoms. The energy necessary to break the directionality of a bond, i.e. to liquefy, is evidently less than the binding energy. It is difficult to estimate just how much less, because the theory of the liquid state is not very well developed.

We simply take ice and water as an example, in which the heat of melting (80 Cal) is about one-seventh of the heat of evaporation. Since the binding energy of ice at zero temperature is somewhat larger than at boiling temperature, it would be a good estimate to assume that the melting energy is about one-tenth of the binding energy. We write

$$E_{liq} = \beta B = \beta \eta Ry ,$$

with $\beta \sim 0.1$ and $\eta \sim 0.2$ for silicon oxide. Substituting for E_{liq} in Eq. (1.5) gives

$$h \le \frac{\beta B'}{A m_p g}$$

or

$$h \le \frac{B \eta Ry}{A m_p g} . \quad (1.6)$$

For silicon dioxide SiO_2, A = 28 + 2(16) = 60 and

$$h \le \beta\eta \times 3 \times 10^8 \text{ cm}$$

$$\le \frac{1}{10} \times \frac{1}{5} \times 2 \times 10^8$$

$$\le 30 \text{ km} ,$$

where $\eta = 1/5$ has been assumed for the solid rock at its melting point.

This shows that a mountain must be less than 30 km high to be supported by the rock at its base. Actually the upper limit is smaller than that because the rock is warm and therefore needs less energy to liquefy. All the mountains we find on Earth are of a height which is of this order or smaller. On other planets the critical height would be different because the acceleration due to gravity g would be different, and the planet may also be made of a different type of material.

Let us eliminate g from this expression because it is not a fundamental constant. This can be done as follows. The force of attraction between a particle of mass m and the Earth mass M_E is

$$mg = \frac{G m M_E}{R_E^2} ,$$

where R_E is the radius of the Earth, i.e.

$$g = \frac{GM_E}{R_E^2} . \quad (1.7)$$

M_E and R_E can be expressed in terms of N_E, the number of nucleons in the Earth -- if we do not look too closely. The Earth consists mostly of silicon dioxide SiO_2 (A = 60) in the crust, and iron (A = 57) in the core. These two substances have approximately the same atomic number and, therefore, size. Their radius can be expressed as fa_0, where $f \sim 4$, so that

$$\text{volume of the Earth} = \frac{4}{3}\pi R_E^3 = \frac{4}{3}\pi \frac{N_E}{A}(fa_0)^3 \quad \text{and} \quad R_E = fa_0\left(\frac{N_E}{A}\right)^{1/3} .$$

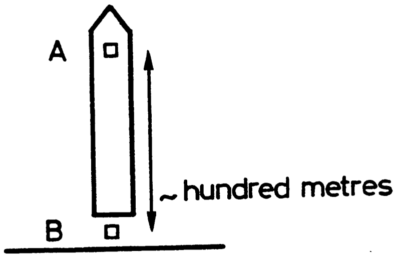
Substituting in Eq. (1.7), we get

$$g = \frac{GM_E}{R_E^2} = \frac{GN_E m_p}{R_E^2}$$

$$= GN_E m_p \times \left(\frac{A}{N_E}\right)^{2/3} \times \frac{1}{(fa_0)^2} . \quad (1.8)$$

G, the gravitational constant, has dimensions. We would like to work with dimensionless quantities. The potential between two electrons -- or any two particles of charge $|e|$, e.g. the proton -- due to Coulomb attraction is e^2/r where r is their separation. The potential between two nucleons due to gravitational attraction is Gm_p^2/r. It can be seen that Gm_p^2 and e^2 play the same role in these expressions. This analogy can be used in defining a fine structure constant of gravity $\alpha_G = Gm_p^2/\hbar c$ (cf. fine structure constant $\alpha = e^2/\hbar c$); α and α_G are dimensionless constants; $\alpha_G = 0.59 \times 10^{-38}$, which is much smaller than the electromagnetic $\alpha = 1/137$, which indicates how weak gravity is.

As an example of how strong the electromagnetic force is, compared with the gravitational force, consider the Apollo rocket which, as we know, is rather big:



If all the electrons were removed from 1 cubic mm of iron in the centre of the rocket at A and placed on the ground below it at B, the force of attraction between these electrons and the residual positive charge at A about a hundred metres away would be sufficient to prevent the rocket from ever leaving the ground, i.e. it is of the same order as the weight of the whole rocket. This example is due to Yuval Ne'eman of Tel-Aviv.

If G is written in terms of α_G in Eq. (1.8), we obtain an expression for g in terms of fundamental constants:

$$g = \frac{\alpha_G \hbar c}{m_p} A^{2/3} N_E^{1/3} \frac{1}{(fa_0)^2} .$$

Substituting g in Eq. (1.6) gives

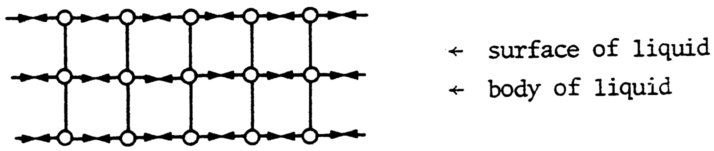
$$\frac{h}{a_0} = \beta \eta f^2 \times \frac{\alpha}{\alpha_G} \times \frac{1}{N_E^{1/3}} \frac{1}{A^{5/3}} ,$$

which is an expression for h/a_0 in terms of dimensionless constants. Only N_E is not a universal constant.

This, of course, gives the same value for h as before with f = 4.

1.8 Number of atoms in a liquid from its surface tension and latent heat of evaporation

Consider a liquid at room temperature. It can be represented in two dimensions by the diagram below



ach atom in the body of the liquid is bound to its neighbours by six bonds (in the thr dimensional case). Therefore the binding energy per atom B = 6b, where b is the strength of a single bond. However, the atoms on the surface have only five bonds producing a resultant force pulling these atoms towards the body of the liquid, which gives the 'skin effect' of surface tension. These atoms therefore have an energy b = B/6 less than other atoms in the liquid. Since b is negative, this results in a positive surface energy. The surface tension S is defined as the surface energy per unit area of surface, which can also be expressed as an atomic energy per atomic area:

$$S = \frac{\text{surface energy per atom}}{\text{area corresponding to 1 atom}}$$

$$= \frac{b}{d^2}$$

$$= \frac{\eta}{6f^2} \frac{Ry}{a_0^2} ,$$

where d is the separation between the nuclei of the atoms.

This formula is very well fulfilled -- within 50%; the error is due mainly to the assumption b = B/6 and depends on the structure of the liquid.

The heat of evaporation per atom is B. Therefore the heat of evaporation per cc

$$E_{ev} = \frac{\text{heat of evaporation per atom}}{\text{volume per atom}}$$

$$= \frac{B}{d^3} .$$

Eliminating B from these two equations we have

$$\frac{S}{E_{ev}} = \frac{1}{6} d$$

$$= \frac{1}{6} \times \frac{1}{N} ,$$

where N is the number of atoms along an edge of length 1 cm. Rearranging, we get

$$N = \frac{1}{6} \frac{E_{ev}}{S} . \tag{1.9}$$

Therefore simply by measuring the surface tension and latent heat of evaporation per unit volume, it is possible to determine the number of atoms in a line 1 cm long. This number cubed gives the number of atoms per cc in the liquid.

The following is a very simple derivation of the same formula. Take 1 cc of water. We would like to count the number of atoms in it. How shall we do this? We take a very sharp knife and cut it into slices one atom thick, i.e. N slices. We repeat this in the other two dimensions. This leaves separated atoms. It is equivalent to the liquid having evaporated. The energy used to separate the atoms in both cases must be the same. Each cut bares 2 square cm of surface and therefore costs 2S in energy. N = 1/d cuts are made in each of three dimensions, therefore the total energy used in cutting up 1 cc of water is

$$2S \times N \times 3 = 6SN \ .$$

This must equal the latent heat of evaporation per cc, E_{ev}, i.e.

$$E_{ev} = 6SN \ ,$$

$$N = \frac{1}{6}\frac{E_{ev}}{S} \ .$$

This is exactly the same relation [Eq. (1.9)] as obtained by the first method.

This is a wonderful way of counting atoms -- wonderful because the number of atoms is terrific, yet it is reduced to two measurable human experiences. We know how much energy it takes to boil away 1 cc of water -- we do it every day. We also have a feeling for how much energy it needs to extend a surface, for example in blowing a soap bubble. Yet together they give us a number so great that we cannot really visualize it.

This should be in every elementary physics textbook, but it can only be found in one little-known book which is otherwise very bad. Written by a (fortunately) completely unknown German physical chemist and entitled 'German Chemistry', it came out at the height of the Nazi régime in Germany in 1938. It contained a lot of chapters that were strongly against quantum mechanics (this being a Western, Jewish invention) and insisted that chemistry should be very much simpler -- and that German chemistry was, quoting this as an example. It shows that one can find pearls wherever one looks for them.

1.9 The Pauli Exclusion Principle

The Pauli Exclusion Principle is usually stated in one of the following ways:

1) It is impossible to have more than one fermion in the same quantum state. For example, it is possible to have two electrons with the same spacial wave function, but one of them must be spin-up, the other one spin-down.

2) The wave functions must be antisymmetric under interchange of the two fermions.

Both these definitions can be very useful; but it would also be very useful to introduce another one which states that particles that obey the Pauli Exclusion Principle, i.e. fermions, must stay apart from each other. We shall show this using the second definition above.

Let us take two electrons of the same spin so that they are really equivalent. Let their position and momentum be x_1, k_1 and x_2, k_2, respectively (for the one-dimensional case). The wave function of the system must be the product of the wave functions of the individual electrons, i.e.

$$\begin{aligned} \psi &= e^{ik_1x_1} e^{ik_2x_2} \\ &= e^{ik(x_1-x_2)} \\ &= e^{ikx} \ , \end{aligned}$$

where $k_1 = -k_2 = k$ is the momentum of each electron in the centre-of-mass frame, and where $x = x_1 - x_2$ is the separation of the two electrons. This equation is non-symmetrized. The correct wave-function must be antisymmetric under interchange of the two electrons, in agreement with the Exclusion Principle, and it is written:

$$\psi = e^{ikx} - e^{-ikx} \ ,$$

where e^{-ikx} is obtained from e^{ikx} by interchange of the labels 1 and 2 of the electrons. This may be written

$$\psi = 2i \sin kx \ .$$

The probability density $|\psi|^2 = -4 \sin^2 kx$. The graph of $|\psi|^2$ against x is shown below (see Fig. 3), i.e. the probability of finding the electrons a certain distance apart as a function of this distance:

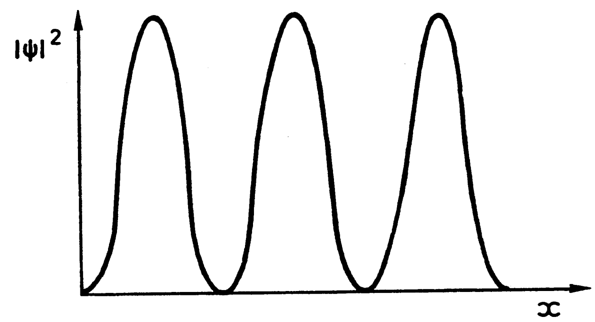
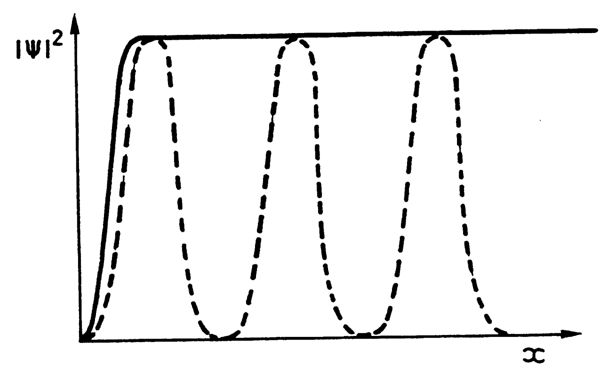


Fig. 3

Let there be a spread in the momentum values of the electrons such that $2/3\ k_0 \leq k \leq 4/3\ k_0$ with a definite average value k_0. $|\psi|^2$ would be an integral of all these waves of different k, such as the one shown above with wavelengths slightly shorter or slightly longer than that of the average corresponding to k_0. This gives a curve as shown below:



Here the solid line represents the resultant and the dotted line is a curve such as in Fig. 3 for the single wave average momentum k_0. This shows that the probability of finding the two electrons a given distance apart is roughly constant at large separations; but as the separation drops to zero, so does the probability. The distance in which this change takes place is approximately the distance in which sin kx goes from 1 to 0, i.e.

in a distance of the order of $0 \leq x \leq 1/k_0$, probably a little smaller. This tells us therefore that two identical fermions must stay apart. Their minimum possible separation is of the order of their average wavelength $\lambdabar = 1/k_0$ corresponding to their momentum $p_0 = \hbar k_0$.

This is a very interesting idea and can be considered to be what is left in quantum theory of the old idea that electrons, protons, and neutrons are hard, impenetrable spheres. This classical idea is of course wrong, but fermions do behave as if they had a certain radius. This radius depends on their momentum and therefore on their energy. The higher the momentum, the smaller is this radius where they effectively become impenetrable.

Let us apply this to an electron gas. Consider N electrons in a volume V, all with parallel spins so that they are identical. Then there is a minimum average energy that these electrons must have. The standard quantum mechanical way of solving this problem is to put the electrons in eigenstates associated with this box of volume V. Each eigenstate can accommodate only one electron. This is called the Fermi gas. The lowest state of the Fermi gas is when the lowest eigenstates are occupied, so that the average energy can be calculated. This is simple enough. But we shall do the calculation another way, which gives the same formula except for the constant which must be obtained from the exact quantum mechanical treatment.

Associated with each electron is a space of volume of the order $(4/3)\pi(\lambdabar/2)^3$, where $\lambdabar$ is related to its momentum through $p = \hbar k = \hbar/\lambdabar$. The volume occupied by all the electrons is

$$V = N \frac{4}{3} \pi \left(\frac{\lambdabar}{2}\right)^3 , \qquad (1.10)$$

where $\lambdabar$ is the maximum possible average value of λ for the given values of N and V. But

$$V = N \times \frac{4}{3} \pi \left(\frac{d}{2}\right)^3 , \qquad (1.11)$$

where d is the mean separation of the electrons, so that

$$\lambdabar \approx d ,$$

i.e. in a Fermi gas the average wavelength of the fermions is of the same order as their mean separation.

For randomly oriented spins, there will be N/2 spin-up and N/2 spin-down particles. Each of these groups may be considered separately since there is no Pauli effect between electrons of opposite spin because they are no longer in the same quantum state, so that Eq. (1.10) becomes

$$V = \frac{N}{2} \times \frac{4}{3} \pi \left(\frac{\lambdabar}{2}\right)^3 .$$

But still

$$V = N \times \frac{4}{3} \pi \left(\frac{d}{2}\right)^3 , \qquad (1.11)$$

so that

$$\frac{\lambda\!\!\!^{-}}{3\sqrt{2}} \approx d ,$$

i.e.

$$\lambda\!\!\!^{-} \approx 3\sqrt{2}\, d \qquad \text{or} \qquad \frac{d}{\lambda\!\!\!^{-}} \approx \frac{1}{3\sqrt{2}} \,^{*)} .$$

We can also use this idea to calculate the average kinetic energy of an electron in a Fermi gas:

$$K_{av} \approx \frac{p^2}{2m_e} = \frac{\hbar^2}{2m_e\lambda\!\!\!^{-2}}$$

$$\approx \frac{\hbar^2}{2m_e d^2} \,^{**)} .$$

1.10 Black-body radiation

Black-body radiation of temperature T is associated with a certain volume V at temperature T containing electromagnetic radiation. It has a definite radiation density, i.e. a definite energy per unit volume, which is given by the Stefan Boltzmann law

$$E_{rad}/cm^3 = \frac{\pi^2}{15}\frac{1}{\hbar^3 c^3}(kT)^4 .$$

This formula can be derived by considering the radiation as being composed of a gas of photons with an average energy of kT. Each photon can be considered as occupying a cube of side $\lambda\!\!\!^{-}$, where $\lambda\!\!\!^{-} = c/\omega$, i.e. they occupy a volume $\approx \lambda\!\!\!^{-3}$. The energy of each photon is $\hbar\omega$, where ω is chosen such that $\hbar\omega = kT$. The energy density is therefore

$$E_{rad}\ cm^{-3} = \frac{\text{energy per photon}}{\text{volume per photon}}$$

$$\approx \frac{\hbar\omega}{\lambda\!\!\!^{-3}}$$

$$= \frac{\hbar\omega}{(c/\omega)^3} = \frac{\hbar\omega^4}{c^3}$$

$$= \frac{(KT)^4}{\hbar^3 c^3} ,$$

which is just the Stefan Boltzmann law.

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*) The result obtained from an exact quantum mechanical treatment for random spin orientation is

$$\frac{d}{\lambda\!\!\!^{-}} = 2.34 .$$

**) The exact figure from quantum mechanics is

$$K_{av} = 5.5\,\frac{\hbar^2}{2m_e d^2}$$

2. NUCLEAR FORCES

Because of its relevance to the study of the stars and their internal mechanisms, we shall say a few words about nuclear physics. So far we have considered only atomic processes. Now it is very simple and instructive to go over from atomic processes to nuclear processes by the following observation. The hydrogen atom is a system of two particles attracted by a Coulomb force. Now the deuteron consists also of two particles, a proton and a neutron, bound together by a different kind of force, namely the nuclear force. Unfortunately it is far more complicated than the Coulomb force, and in fact we do not really understand it. Figure 4 shows a plot of the nuclear potential of the deuteron as a function of the radius r:

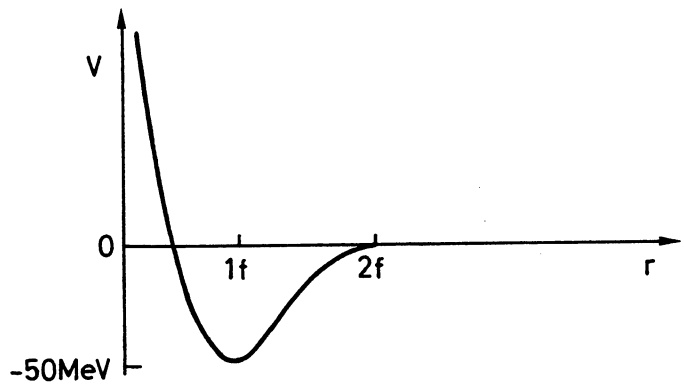

Fig. 4

The r scale is calibrated in fermis, where 1 fermi = 10^{-13} cm. The precise form of V(r) depends on whether the spins of the nucleons are parallel or antiparallel and whether their relative quantum state is symmetric or antisymmetric, but broadly speaking it has three main features:

i) it is repulsive at small distances, up to about $\frac{1}{2}$ fermi;
ii) it is attractive for distances over about 1 fermi;
iii) it has a finite range of about 2 fermis.

Now if you know some chemistry you will notice that V(r) resembles the chemical potential between two atoms in a molecule, which is also repulsive at small distances and attractive at large distances. This may be significant. The chemical force is not after all a fundamental force but rather a very complicated left-over from the Coulomb attractions between the constituents of the atoms. It could well be, then, that the nuclear force is also a derived kind of effect such as a left-over force from interactions between more fundamental particles -- perhaps the quarks, if they exist. At present, of course, much experimental work is being done to determine the origin of the nuclear force. One recalls that the discoveries of Rutherford and Bohr as to atomic structure gave a basis from which to derive the complicated chemical forces. We have not yet reached that stage in regard to the nucleon structure, and this is why we do not know the basis of the nuclear force.

Comparing the nuclear force with the Coulomb force reveals two fundamental differences namely that it is repulsive at short distances and has a finite range. However, to get a rough quantitative comparison between the two forces we can say that in the effective attractive region of the nuclear force it is about ten times as strong as the Coulomb attraction. In other words, in that region the nuclear potential goes roughly as g^2/r, where $g \sim 3.3\ e$. Let us therefore use this to get a rough order of magnitude idea of the radius and the binding energy of the deuteron. Earlier we derived the corresponding formulae for the hydrogen atom, namely:

$$a_0 = \hbar^2/m_e e^2 \qquad \text{and} \qquad E_0 = -m_e e^4/\hbar^2 \ .$$

We replace e by g and m_e by m_p, the proton mass, and immediately obtain for the deuteron:

$$a_D = \hbar^2/m_p g^2 \sim a_0/20{,}000 \ ,$$

which is of the order of a few fermis,

and
$$E_D = -m_p g^4/\hbar^2 \sim 200{,}000\ E_0 \ ,$$

which is around a few million electron volts.

Of course one has to be a little careful because of the differences between the two systems. The fact that the deuteron exists only in the ground state $(L = 0)$ (apart from the 5% admixture of $L = 2$ from the fact that the system is not quite spherically symmetric) is a consequence of the short-range nature of the force. Suppose, for example, the deuteron existed in the state $L = 1$. Then by analogy with the hydrogen atom it would have a radius of $4\ a_D$, and this is outside the range of the nuclear force. Consequently the proton and neutron would fly apart.

This situation is characteristic of the difference between the nuclear and Coulomb forces. However, we shall use the values a_D and E_D to give a rough measure of nuclear systems until the day, perhaps in twenty years time, when we shall have a coherent theory at our disposal.

3. THE STARS*)

3.1 Introduction

We shall now give an estimate of the size of the stars and shall discuss their manner of evolution. We shall see that some basic constants, some characteristic magnitudes that we have already shown to be fundamental in atomic, molecular, and nuclear phenomena, will enter into our discussion. To simplify the discussion we shall assume that a star has constant density. This of course is false, the density increasing towards the interior, but we shall talk in terms of an average density and an average radius. It turns out that these errors are smaller than they would appear, because a star's density decreases rather rapidly in its outer layers, and consequently we get answers of the correct orders of magnitude. Of course strictly speaking this is an unscientific approach, but for the sake of clarity it is fun and at the same time instructive to talk in this way after the astrophysicists have solved the complicated differential equations.

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*) The following considerations were inspired by reading E. Salpeter's article in Bethe Festschrift: "Perspectives in Modern Physics", edited by R. Marshak, 1966.

3.2 The Virial Theorem -- Forces at work in stars

We first derive for stars the well-known Virial Theorem, which relates the internal kinetic energy to the gravitational potential energy. We shall assume that a star consists partly of hydrogen at high temperature, or in other words free protons and electrons, and partly of radiation, i.e. photons, and shall ignore a 1% to 2% contribution from other nuclei.

Two opposing forces keep a star in equilibrium. First there is gravity, which tends to contract the star, and secondly there is the pressure P that resists this contraction. This pressure arises from the kinetic energy of the constituent particles, rather like molecules in a gas.

Now for a star of constant density the gravitational potential takes the form $\Omega = -\lambda GM^2/R$, where R and M are the radius and mass of the star, G is the gravitational constant, and λ some numerical factor of the order unity. If we increase the radius by a small amount dR and hence the volume by dV, the increase in energy U of the star will be given by

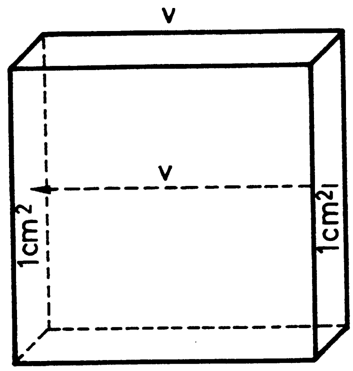
$$dU = -PdV + d\Omega \ .$$

If the star is in stable equilibrium its energy will be minimal, and consequently we must have dU = 0. Using $V \propto R^3$ gives:

$$0 = -3PV\ dR/R + \lambda GM^2/R^2\ dR \ ,$$

and so we obtain

$$3PV = \lambda GM^2/R = -\Omega \ .$$



Next let us consider the form of the pressure P. For particles in a perfect gas $P = \frac{1}{3}\, n\, pv$, where n is the density of particles and pv is the mean value of the product momentum times velocity for each. To see this we need only remark that a particle of momentum p striking a wall normal to its path and suffering elastic reflection transmits to the wall a momentum 2p. If there are n such particles per cubic centimetre, then the number striking 1 cm² of wall per second will be nv. The contribution to the total pressure by these particles will then be 2n pv. When we add up over all particles we get a factor $\frac{1}{6}$, since only one-sixth of particles with speed v will strike the wall (there are six directions in space). For non-relativistic particles we can use p = mv to get

$$P = \frac{1}{3} mn\ v^2 = \frac{2}{3}\varepsilon \ ,$$

where ε is the mean kinetic energy density (i.e. per unit volume).

On the other hand, for relativistic particles $v \sim c$, and so we must use p = E/c and this gives

$$P = \frac{1}{3} n\ E = \frac{1}{3}\varepsilon \ .$$

This last relation is of course exact for photons. For once, the factor 2 which differentiates our two expressions for P is significant.

Writing $V\varepsilon = K$, the internal kinetic energy of the star, we get $2K = -\Omega$ for non-relativistic particles and $K \to -\Omega$ for highly relativistic particles. So the total energy $U = K + \Omega$ is given by $U = \frac{1}{2}\Omega$ for non-relativistic particles and $U \to 0$ for highly relativistic particles. The difference, as we have said, is significant. In the first case U is negative since Ω is, and so it follows that the star is stable. In the second case U is zero, or strictly speaking small and negative, so that no energy or rather little energy is needed to take the star apart. Consequently, highly relativistic stars are unstable. The reason for this physically is that the radiation pressure within a star is far more effective than the particle pressure in overcoming the gravitational forces.

3.3 Cool stars

3.3.1 Size and temperature

Let us first consider cool stars; in other words, stars for which the radiation pressure is small compared with the non-relativistic particle pressure. So we consider only protons and electrons, and let there be N of each. If the temperature of the star is T then the mean kinetic energy per particle is $\frac{3}{2}$ kT, where k is Boltzmann's constant.

So the total kinetic energy of the star $K = 3NkT$, i.e.

$$3NkT = -\frac{1}{2}\,\Omega = \frac{1}{2}\,\lambda\; GM^2/R \;.$$

Dropping numerical factors of the order unity and dividing by N, we then get for the energy per particle

$$kT \sim GM^2/NR \;.$$

Now let d be the average distance between neighbouring protons so that $Nd^3 \sim R^3$. Further, putting $M = N(m_p + m_e) \sim Nm_p$ (neglecting the electron mass compared with the proton mass), we obtain

$$kT \sim Gm_p^2\; N^{2/3}/d \;.$$

With the "fine structure constant" for gravity defined before by $\alpha_G = Gm_p^2/\hbar c$ and a number N_0 defined as $N_0 = (1/\alpha_G)^{3/2}$ we get

$$kT \sim (N/N_0)^{2/3}\; \hbar c/d \;. \qquad (3.1)$$

We shall see that N_0 turns out to be about the number of protons in the Sun!

3.3.2 Evolution of a cool star

Let us first, however, use our result (3.1) to describe the evolution of a star. Initially when the star is being formed from a contracting cloud of hydrogen, d is very large and so T is very small. As the star contracts, d becomes smaller and T becomes larger. Furthermore, from the result $U = \frac{1}{2}\Omega \sim -GM^2/R$ we know that the total energy of the star decreases as it contracts. In fact, as it contracts it gets hotter and hotter and consequently radiates energy. So we have a system which is losing energy and yet whose temperature is actually increasing!

Of course this is a consequence of the Virial Theorem which tells us that the star loses more in potential energy than it gains in internal kinetic energy. A similar effect can be seen in the case of the motion of the Moon around the Earth. Suppose that we tried to reduce the speed of the Moon by hitting it with a large rocket in the opposite direction to its motion. After the drastic blow its energy would be decreased and it would start to fall towards the Earth. Eventually it would take up an orbit nearer to the Earth, and in that case its orbital velocity would actually be greater than before. So if you try to brake the Moon's speed it goes faster.

3.3.3 Electrons want space; Pauli supports them

Now we have our star contracting, getting hotter and radiating, but the process of this collapse does not continue indefinitely. The limit is in fact given by the compressibility of matter, which in turn is explained by the Pauli Principle. Recall that this principle implied that the kinetic energy of electrons in some given volume must be at least $\hbar^2/m_e d^2$ per electron, where d is their mean distance apart. (Since the number of protons and electrons is the same, this d is the same as before.) The corresponding pressure is called the degeneration pressure, and physically it arises from the cushioning effect of the electrons. The corresponding effect for protons can be neglected because $1/m_p$ is very much smaller than $1/m_e$.

Previously we neglected the degeneration pressure and wrote only $P = \frac{1}{3}\varepsilon$, so that $PV = \frac{1}{3}K = NkT$. We can modify this in a semiquantitative way by rewriting

$$PV = NkT + \frac{N\hbar^2}{m_e d^2} .$$

Equation (3.1) then becomes:

$$kT + \frac{\hbar^2}{m_e d^2} \sim \left(\frac{N}{N_0}\right)^{2/3} \frac{\hbar c}{d} . \qquad (3.2)$$

3.3.4 Maximum temperature. Minimum size

Although the correct expression is more complicated than this, it has a reasonably good asymptotic form for small and large d. The plot of -kT against d is shown in Fig. 5. The shape of the curve is again typical of the form $A/d^2 - B/d$, like the atomic energy in Section 1.1.

It follows that the consequence of the degeneration pressure term is twofold:

i) The star has a maximum possible temperature depending upon its size and this is given by $kT_{max} = (N/N_0)^{4/3}\, m_e c^2$.

ii) The star contracts to a minimum radius for which T = 0. Smaller radii would lead to negative temperatures, which are not physically possible. It is then a mere cold chunk of matter $d_{min} = \hbar/m_e c\, (N_0/N)^{2/3}$. When going down from T_{max} to T = 0, it radiates a lot within a very short time. During this stage, stars are called "white dwarfs". The value of d for T_{max} is in fact about $2d_{min}$.

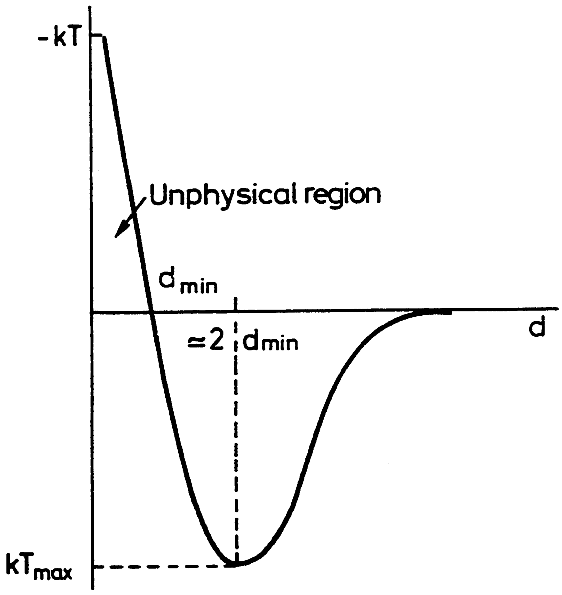


Fig. 5

3.3.5 The nuclear reactions and the qualified stars

And now, using the expression for T_{max}, we shall derive the minimum allowed size of a star. We require T_{max} to be high enough for nuclear reactions to take place, otherwise by definition we do not have a star. If the temperature is always too low, the system collapses, becomes hotter, and then cools down (see Fig. 5), the whole process taking about a million years, whereas a star in which nuclear reactions take place lives for more like a billion.

Nuclear reactions take place if the energies involved are of the order of one-tenth MeV and by pure chance the electron mass is of this order of magnitude, being around 0.5 MeV. So let us suppose that nuclear reactions will start at mean temperatures somewhat smaller than this limit, i.e. at fm_ec^2, where f is of the order of $1/10$ to $1/50$. Then we require that

$$kT_{max} > fm_ec^2 \quad ,$$

i.e. $N/N_0 > f^{3/4} \sim 1/10$.

Now N_0 is of the order of 10^{57}, which (to within a factor 2) is about the number of protons in the Sun, and so our relation shows that the smallest stars have masses of around one-tenth of that of the Sun. Indeed no smaller stars that really shine have ever been observed.

Systems below the limit $1/10\ N_0$ do not deserve the name of stars. Their history is determined only by the opposite actions of pressure and gravity. The latter is not strong enough to overcome the cushion effect of the electrons and to raise the temperature to the critical value where nuclear reactions can take place, thus extending the lifetime of the star. Such a system, after a highly radiative final stage (white dwarf stage) dies into a cold chunk of matter. If N/N_0 is not too small, this final state will be highly compressed. So let us ask what is the largest value of N for which such a cold object will not be too compressed?

3.3.6 When matter does not become a star

Now for matter with which we are familiar, the distance apart of neighbouring protons is bigger than the Bohr radius. So let us demand:

$$d_{min} > a_0 = \hbar^2/m_e c^2 \ .$$

This tells us that

$$\hbar/m_e c \ (N_0/N)^{2/3} > \hbar^2/m_e c^2 \ .$$

In other words,

$$N/N_0 < (e^2/\hbar c)^{3/2} = \alpha^{3/2} \sim 1/1000 \ .$$

So for matter as we know it, in which atoms have electron orbits, we must have

$$N < N_0/1000 \ .$$

The mass of Jupiter is typical for this limit: so what we have really found is an upper limit for the sizes of planets.

We can find a lower limit for the sizes of planets by demanding that the heights of mountains are smaller than the radii! A planet must after all be reasonably round. It therefore must be large enough in order that gravity is able to overcome the forces of rigidity if it has an odd shape at the outset. From our previous discussion on the height of mountains, one gets for the lower limit R* of the radius of a planet

$$(R^*)^2 = \frac{\beta\eta f^3\alpha}{A^2\alpha_G} a_0^2 \ .$$

The minimum radius for a planet is around 500 kilometres. This figure is just the upper limit for the radii of asteroids.

Let us briefly summarize. By stipulating, on the one hand, that matter should not be too compressed, and on the other that the heights of mountains should not be too large, we have found upper and lower limits for the sizes of planets. The Earth is just between those two limits. Again by stipulating that nuclear reactions must take place, we have found a lower limit for the sizes of stars.

3.4 Hot stars

We shall now derive the upper limit for the size of a star. Big stars become hot and radiation can no longer be neglected. We recall from the Virial Theorem that a star consisting mostly of radiation will be unstable, and it is this instability that will provide the upper limit required.

3.4.1 The photons come in

We shall use Boltzmann's law which says that the energy density of photons at temperature T is about $(kT)^4/\hbar^3c^3$. Recalling that

$$P = \frac{2}{3}\varepsilon \text{ for non-relativistic matter}$$

or $\frac{1}{3}\, \varepsilon$ for photons,

we have $PV = NkT + (kT)^4\, V/\hbar^3c^3$, dropping numerical constants of the order unity. Equation (3.1) modified by the radiation term then becomes

$$kT + (kT)^4\, d^3/\hbar^3c^3 \sim (N/N_0)^{2/3}\, \hbar c/d \ , \tag{3.3}$$

where we have used the fact that $Nd^3 \sim V$. Writing $x = kT\, d/\hbar c$ gives us the simple relation

$$x(1 + x^3) \sim (N/N_0)^{2/3} \ .$$

The question we must now ask is: "When will the radiation content dominate the matter content?"; for this, after all, is our criterion for instability. The answer is: "When x^3 is large compared to 1", and this is true only if N/N_0 is large. In fact our condition for instability is $N/N_0 > 50$, say, and this therefore gives the upper limit for the sizes of stars.

Remembering that $N_0 \sim 10^{57}$ we have shown that the number of protons in a star must lie roughly in the region 10^{56} to 10^{59}.

3.4.2 But electrons also are now relativistic

So far we have worked on the assumption that the protons and electrons in the star were non-relativistic. For a cool star we had $kT_{max} = (N/N_0)^{4/3}\, m_ec^2$, and so for $N/N_0 \sim 1/5$, say, the kinetic energies of the particles are small compared with their rest mass energies. Our assumption was then correct. For hot stars, however, with $N/N_0 > 1$, this is no longer true. The electrons go faster and faster, and relativistic electron-positron pairs become more and more abundant as the photon energy also becomes larger than m_ec^2. Now the degeneration pressure $\hbar^2/m_ed^2$ was calculated on the basis that the wavelength of an electron $\lambda \sim d$ (Pauli principle) and that its energy in the non-relativistic limit was $p^2/2m_e$, where $p = \hbar/\lambda$. In other words, for relativistic electrons we must replace $\hbar^2/m_ed^2$ by $\hbar c/d$. Consequently, the cushion effect, which previously arose because of the $1/d^2$ term, now disappears. Furthermore, in a hot star the term $\hbar c/d$, which should have appeared on the left-hand side of Eq. (3.3) is small compared with the right-hand side because $N/N_0 > 1$. This is why we neglected it.

So a hot star collapses and gets hotter, becoming more and more relativistic, and there is no cushion effect to stop this collapse. Of course the process is held up by the nuclear reactions which produce energy as fast as the star radiates it, but when the nuclear fuel is exhausted the collapse continues (see Fig. 6):

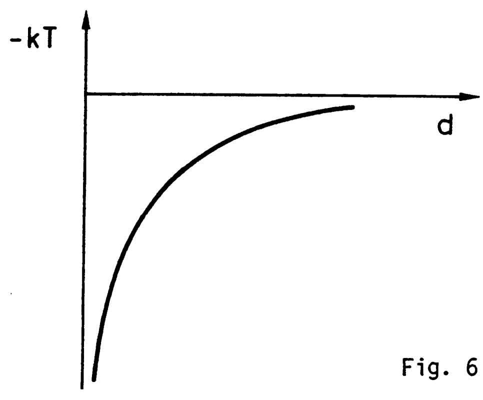


Fig. 6

3.4.3 Becoming neutrons

With increasing temperatures and decreasing interparticle distances, a new phenomenon appears which we shall call the inverse neutron decay, although it is not just the inverted reaction of the decay of the neutron. It is well known that free neutrons decay into protons, electrons, and antineutrinos according to

$$n \to p + e^- + \bar{\nu}$$

with an energy output of 0.8 MeV due to the mass difference between the mass of the neutron and the masses of the proton and the electron together. The "inverse neutron decay" that we refer to is the reaction

$$e^- + p \to n + \nu - 0.8 \text{ MeV} ,$$

which becomes possible when electrons have kinetic energies of the order of 1 MeV or more. The reaction is then even favoured as it absorbs part of the kinetic energy of the electrons.

We then have a neutron star. Initially the neutrons are non-relativistic because they have a higher mass than the electrons, and consequently there is a small cushion effect. Soon, however, T rises still further, the neutrons become relativistic, the star continues collapsing, and there is no source of pressure that can stop this.

3.4.4 The pulsars and cosmic rays

Virtually two things can happen, one pleasant and the other unpleasant.

The pleasant thing

Stellar objects usually rotate, and the conservation of angular momentum during the contraction process requires that the angular velocity should increase. Any initial asymmetries in the shape of the star would be enhanced by this process. Sooner or later the star would break up into fragments, and the chances are that for each one we would have $N < N_0$. For each part our analysis for the cool stars would then apply, except that the cushioning effect would be supplied by the neutrons. The radii of these fragments would finally be very small -- around 10 km -- and their rotational periods, assuming a period of around 25 days for the parent star*), would be of the order of milliseconds. Any radiation emitted from such an asymmetric entity would cause it to flicker, and this we believe is the origin of the pulsars. Furthermore, the magnetic field of such a star would be pulled in and hence greatly magnified by the collapse, and this would enable escaping particles to be accelerated up to incredible speeds. This, we believe, is the origin of the cosmic rays.

3.4.5 Gravitation makes it black

The unpleasant thing

Suppose that the collapsing hot star never divides. Then there is no limit to the collapse, but there is a limit to our knowledge of what actually happens if we are prepared to watch only from afar. This limit is given by the Schwarzschild radius. If a star

*) This is about the period of the sun.

collapses beyond the Schwarzschild radius terrible things happen, but fortunately we never see them. So what you do not see you do not worry about.

Consider a photon of frequency ω and energy $\hbar\omega$ leaving the surface of the star. Let us calculate the radius of the star for which the gravitational field is so strong that the photon never escapes but always falls back to the surface.

The potential energy of the photon at the surface is

$$-\frac{GM}{R}\frac{\hbar\omega}{c^2}$$

and so its kinetic energy at infinity would be given by

$$\hbar\omega' = \hbar\omega - \frac{GM}{R}\frac{\hbar\omega}{c^2} \; .$$

Consequently, as soon as $R = GM/c^2$, $\hbar\omega' = 0$, the photon never escapes and we never see what happens. GM/c^2 is called the Schwarzschild radius. Actually, General Relativity predicts a value precisely double this, but we are interested only in orders of magnitude. Notice, in passing, that the gravitational potential energy of a particle of mass m at the Schwarzschild radius is $-GMm/R = -mc^2$, whereas its rest mass energy is mc^2. So its total energy is zero. So particle creation at the Schwarzschild radius would appear not to contradict the conservation of energy!

We have seen that the final collapse of a hot star is never seen by the outside observer, although this is not true from the point of view of an observer collapsing in with the star. We can put this another way by saying that the post-Schwarzschild collapse takes place in the "more than infinite future" of the outside observer.

3.4.6 Where high-energy physics could give a hand to astrophysics

We end with the following brief remarks. On Earth, atomic physics is important, the temperatures being in the electron volt regions. In the centres of the stars, nuclear physics is important because the temperatures there are in the million electron volts region. We now ask: "Where in the universe is high-energy physics at home in the sense that it is the main process?" The answer lies in the fact that the only source available for producing bulk energies is gravity. High-energy physics is characterized by the fact that the excitation energies of the particles are of the same orders of magnitude as their rest mass energies, i.e. of the order of 1 GeV. High-energy phenomena will occur when particles have kinetic energies comparable to their rest-masses. The Virial Theorem tells us that the kinetic energies of particles are of the same orders of magnitude as their gravitational potential energies, and we have just seen that these are equal to their mass energies just at the Schwarzschild limit. (It follows that high-energy physics is important in a star just at that time when you cannot see it!)

The exciting possibility arises that high-energy phenomena in bulk are really closely connected with gravitational phenomena, thus providing a tool for our investigation of the extreme conditions in stellar evolution. There is the link of the infinitely small to the infinitely big.