Advanced Quantum Physics: Problem Set 3

1 Semiclassics

Section A: mostly bookwork.

A.1 Explain why the action for a particle in a simple harmonic oscillator, between times $t = 0$ and $t = T$, is given by

$$S[x] = \frac{1}{2}m\int_0^T \mathrm{d}t' \left(\dot{x}^2 - \omega^2 x^2\right). \tag{1}$$

[2 marks]

A.2 Show that classical trajectories obey

$$\ddot{x} = -\omega^2 x.$$

[5 marks]

A.3 Solve for the classical trajectory $x(t)$ assuming $x(t=0) = 0$ and $x(t=T) = X$.

[3 marks]

Section B: bringing together ideas from across the course.

B.1 Explain the method of stationary phase.

[2 marks]

B.2 Find an expression for the normalisation $\mathcal{Z}^{-1}$ in terms of a Gaussian functional integral (which you do not need to evaluate).

[3 marks]

B.3 Using the method of stationary phase and your answer to **A.3**, find an approximate expression for the propagator of the harmonic oscillator. You do not need to evaluate the normalisation $\mathcal{Z}$.

[5 marks]

Section C: more challenging.

Now consider the harmonic oscillator forced with a time-independent force F, for which the action is

$$S[x] = \int_0^T \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 - Fx\right)\mathrm{d}t'. \tag{2}$$

C.1 Show that the propagator is

$$K_{forced}(X, T; 0, 0) = \exp\left(i\epsilon T/\hbar\right) K(X, T; 0, 0)$$

where ϵ is an energy you should find.

[5 marks]

Semiclassics (worked example solutions)

Section A: mostly bookwork.

A.1 Explain why the action for a particle in a simple harmonic oscillator, between times $t = 0$ and $t = T$, is given by

$$S[x] = \frac{1}{2} m \int_0^T \mathrm{d}t' \left(\dot{x}^2 - \omega^2 x^2\right). \tag{3}$$

[2 marks]

The action in general is

$$S = \int_0^T \mathrm{d}t' L(x, \dot{x}) = \int_0^T \mathrm{d}t' \left(\frac{1}{2} m \dot{x}^2 - V(x)\right) \tag{4}$$

(excluding magnetic fields, curved spaces, and so on, that we haven't met yet and which are irrelevant here).
[1 mark]
The harmonic oscillator potential is

$$V(x) = \frac{1}{2} m \omega^2 x^2$$

[1 mark]
giving the desired result.

A.2 Show that classical trajectories obey

$$\ddot{x} = -\omega^2 x.$$

[5 marks]

Classical trajectories are those for which the action is stationary.
[1 mark]

$$S[x] = \frac{1}{2} m \int_0^T \mathrm{d}t' \left(\dot{x}^2 - \omega^2 x^2\right)$$

I will use functional derivatives, but any method is fine. Other choices are to explicitly vary the trajectory $x \to x + \lambda \epsilon$, or to quote the general Euler Lagrange equation and insert the Lagrangian.

$$\frac{\delta S}{\delta x(t)} = \frac{1}{2} m \int_0^T \mathrm{d}t' \left(2\dot{x} \frac{\delta \dot{x}(t')}{\delta x(t)} - 2\omega^2 x \frac{\delta x(t')}{\delta x(t)}\right)$$

[1 mark]
and integrate the $\dot{x}$ term by parts:

$$\frac{\delta S}{\delta x(t)} = \frac{1}{2} m \int_0^T \mathrm{d}t' \left(-2\ddot{x} \frac{\delta x(t')}{\delta x(t)} - 2\omega^2 x \frac{\delta x(t')}{\delta x(t)}\right)$$

(the boundary term vanishes by construction, as usual).
[1 mark]
Now use

$$\frac{\delta x(t')}{\delta x(t)} = \delta(t - t')$$

to give

$$\frac{\delta S}{\delta x(t)} = -m \int_0^T \mathrm{d}t' \delta(t-t')\left(\ddot{x}+\omega^2 x\right)$$
$$= -m\left(\ddot{x}+\omega^2 x\right)$$

[1 mark]
where the time variable in the last line is t (as opposed to t' inside the integral). Set this to zero
[1 mark]
to find

$$\ddot{x} = -\omega^2 x.$$

A.3 Solve for the classical trajectory $x(t)$ assuming $x(t=0)=0$ and $x(t=T)=X$.

[3 marks]

The general solution to the equation in A.2 is

$$x(t) = A\exp(i\omega t) + B\exp(-i\omega t).$$

[1 mark]
We require that

$$x(0) = 0$$

giving

$$x(t) = A\sin(\omega t)$$

[1 mark]
and

$$x(T) = X$$

giving

$$x(t) = X\frac{\sin(\omega t)}{\sin(\omega T)}. \tag{5}$$

[1 mark]

Section B: bringing together ideas from across the course.

B.1 Explain the method of stationary phase.

[2 marks]

The propagator is a functional integral over all possible trajectories, weighted by a phase $\exp(iS/\hbar)$.
[1 mark]
The method of stationary phase acknowledges that the biggest contribution to the integral comes from those paths near the classical paths, for which the variation of S is zero.
[1 mark]

For a propagator

$$K(x,t;x_0,t_0) = \int \mathcal{D}x \exp(iS[x]/\hbar) \tag{6}$$

the method of stationary phase gives

$$K(x,t;x_0,t_0) \approx \mathcal{Z}^{-1} \exp\left(iS[x_c]/\hbar\right)$$

where $x_c(t)$ is a classical trajectory.

B.2 Find an expression for the normalisation $\mathcal{Z}^{-1}$ in terms of a Gaussian functional integral (which you do not need to evaluate).

[3 marks]

Expand the action around the classical trajectory using a Taylor series:

$$S[x] = S[x_c] + (x - x_c) \left.\frac{\delta S}{\delta x}\right|_{x=x_c} + \frac{1}{2}(x - x_c)^2 \left.\frac{\delta^2 S}{\delta x^2}\right|_{x=x_c} + \ldots$$

[1 mark]
and note that, by definition, the second term vanishes for classical trajectories, so

$$S[x] \approx S[x_c] + \frac{1}{2}(x - x_c)^2 \left.\frac{\delta^2 S}{\delta x^2}\right|_{x=x_c}$$

[1 mark]
therefore

$$K(X,T;0,0) \approx \exp\left(iS[x_c]/\hbar\right) \int \mathcal{D}x \exp\left(\frac{i}{2\hbar}(x - x_c)^2 \left.\frac{\delta^2 S}{\delta x^2}\right|_{x=x_c}\right).$$

A change of variables will remove the x_c term to give the simple Gaussian functional integral

$$K(X,T;0,0) \approx \exp\left(iS[x_c]/\hbar\right) \int \mathcal{D}x \exp\left(\frac{i}{2\hbar}x^2 \left.\frac{\delta^2 S}{\delta x^2}\right|_{x=x_c}\right).$$

The normalisation is therefore

$$\mathcal{Z}^{-1} = \int \mathcal{D}x \exp\left(\frac{i}{2\hbar}x^2 \left.\frac{\delta^2 S}{\delta x^2}\right|_{x=x_c}\right).$$

[1 mark]

B.3 Using the method of stationary phase and your answer to **A.3**, find an approximate expression for the propagator of the harmonic oscillator. You do not need to evaluate the normalisation $\mathcal{Z}$.

[4 marks]

$$\begin{aligned}
S[x_c] &= \frac{1}{2}m \int_0^T \mathrm{d}t' \left(\dot{x}^2 - \omega^2 x^2\right) \\
&= \frac{m\omega^2 X^2}{\sin^2(\omega T)} \int_0^T \mathrm{d}t' \sin^2(\omega t') \\
&= \frac{m\omega^2 X^2}{\sin^2(\omega T)} \int_0^T \mathrm{d}t' \frac{1 - \cos(2\omega t')}{2} \\
&= \frac{1}{2}\frac{m\omega^2 X^2}{\sin^2(\omega T)} \left[T - \frac{\sin(2\omega T)}{2\omega}\right]
\end{aligned}$$

[4 marks]
So finally

$$K\left(X,T;0,0\right)=\mathcal{Z}^{-1}\exp\left(i\frac{m\omega^2X^2}{2\hbar\sin^2\left(\omega T\right)}\left[T-\frac{\sin\left(2\omega T\right)}{2\omega}\right]\right).$$

[1 mark]

Section C: more challenging.

Now consider the harmonic oscillator forced with a time-independent force F, for which the action is

$$S\left[x\right]=\int_0^T\left(\frac{1}{2}m\dot{x}^2-\frac{1}{2}m\omega^2x^2-Fx\right)\mathrm{d}t'. \tag{7}$$

C.1 Show that the propagator is

$$K_{forced}\left(X,T;0,0\right)=\exp\left(i\epsilon T/\hbar\right)K\left(X,T;0,0\right)$$

where ϵ is an energy you should find.

[5 marks]

We can complete the square on the potential terms:
[1 mark]

$$K_{forced}\left(X,T;0,0\right)=\int\mathcal{D}x\exp\left(i\int_0^T\left(\frac{1}{2}m\dot{x}^2-\frac{1}{2}m\omega^2\left(x^2-\frac{2}{m\omega^2}Fx\right)\right)\mathrm{d}t'/\hbar\right)$$

$$K_{forced}\left(X,T;0,0\right)=\int\mathcal{D}x\exp\left(i\int_0^T\left(\frac{1}{2}m\dot{x}^2-\frac{1}{2}m\omega^2\left(x-\frac{1}{m\omega^2}F\right)^2+\frac{1}{2m\omega^2}F^2\right)\mathrm{d}t'/\hbar\right)$$

[1 mark]
and the remaining F^2 term pulls out of the functional integral:

$$K_{forced}\left(X,T;0,0\right)=\exp\left(i\int_0^T\frac{1}{2m\omega^2}F^2\mathrm{d}t'/\hbar\right)\int\mathcal{D}x\exp\left(i\int_0^T\left(\frac{1}{2}m\dot{x}^2-\frac{1}{2}m\omega^2\left(x-\frac{1}{m\omega^2}F\right)^2\right)\mathrm{d}t'/\hbar\right)$$

[1 mark].
Finally notice that we can change variables in the functional integral to

$$y\left(t\right)=x\left(t\right)-\frac{1}{m\omega^2}F$$
$$\dot{y}\left(t\right)=\dot{x}\left(t\right)$$

giving

$$K_{forced}\left(X,T;0,0\right)=\exp\left(i\int_0^T\frac{1}{2m\omega^2}F^2\mathrm{d}t'/\hbar\right)\int\mathcal{D}y\exp\left(i\int_0^T\left(\frac{1}{2}m\dot{y}^2-\frac{1}{2}m\omega^2y^2\right)\mathrm{d}t'/\hbar\right)$$
$$=\exp\left(i\int_0^T\frac{1}{2m\omega^2}F^2\mathrm{d}t'/\hbar\right)K\left(X,T;0,0\right)$$

[1 mark]
and finally the integral is trivial as F is constant:

$$K_{forced}\left(X,T;0,0\right) = \exp\left(i\frac{F^2T}{2m\hbar\omega^2}\right)K\left(X,T;0,0\right)$$

giving

$$\epsilon = \frac{F^2}{2m\omega^2}$$

[1 mark]

2 Semiclassics

In this question we will apply the WKB approximation to the harmonic oscillator.

Section A: mostly bookwork.

A.1 Derive the WKB approximation

$$\psi(x) = \frac{A_+}{\sqrt{p(x)}} \exp\left(i \int p(x)\,\mathrm{d}x/\hbar\right) - \frac{A_-}{\sqrt{p(x)}} \exp\left(-i \int p(x)\,\mathrm{d}x/\hbar\right) \tag{8}$$

[7 marks]

A.2 The energy of the classical harmonic oscillator is

$$E = \frac{p(x)^2}{2m} + \frac{1}{2} m\omega^2 x^2. \tag{9}$$

Identify the two classical turning points $x_\pm$ of the harmonic oscillator potential.

[3 marks]

Section B: bringing together ideas from across the course.

B.1 The classical turning points of the harmonic oscillator are 'soft'. Explain how this leads to the Bohr Sommerfeld quantization condition

$$\oint p\mathrm{d}x = 2\pi\hbar\left(n + \frac{1}{2}\right). \tag{10}$$

[2 marks]

B.2 Using the Bohr Sommerfeld quantization condition, find the WKB approximation for energy of the quantum particle in the harmonic oscillator.

[6 marks]

B.3 Explain for which energies the WKB approximation to the wavefunction will be the most accurate.

[2 marks]

Section C: more challenging.

C.1 Explain with the aid of a diagram the possible trajectories undertaken by the quantum particle through phase space (x, p), including the areas enclosed.

[5 marks]

Answers to Q2

In this question we will apply the WKB approximation to the harmonic oscillator.

Section A: mostly bookwork.

A.1 Derive the WKB approximation

$$\psi(x) = \frac{A_+}{\sqrt{p(x)}} \exp\left(i \int p(x)\, \mathrm{d}x/\hbar\right) - \frac{A_-}{\sqrt{p(x)}} \exp\left(-i \int p(x)\, \mathrm{d}x/\hbar\right) \tag{11}$$

[7 marks]

This is straight from the notes, so I will not write out the solution. But you should work through it at least once!

The energy of the classical harmonic oscillator is

$$E = \frac{p(x)^2}{2m} + \frac{1}{2} m\omega^2 x^2. \tag{12}$$

A.2 Identify the two classical turning points $x_\pm$ of the harmonic oscillator potential.

[3 marks]

The classical turning points occur when $E = V(x) = \frac{1}{2} m\omega^2 x^2$.
[1 mark]
Therefore

$$x_\pm = \pm\sqrt{\frac{2E}{m\omega^2}}$$

[2 marks]

Section B: bringing together ideas from across the course.

B.1 The classical turning points of the harmonic oscillator are 'soft'. Explain how this leads to the Bohr Sommerfeld quantization condition

$$\oint p \mathrm{d}x = 2\pi\hbar \left(n + \frac{1}{2}\right). \tag{13}$$

[2 marks]

For soft turning points, we need to add a $\pi/2$ phase shift for each classical turning point in the cycle. I don't have a good analogy for this; it comes from Morse theory, and is called the Maslov index. It is complicated, and its consideration led to a major result in algebraic geometry / string theory (Witten's conjecture). But it's worth remembering. Hard boundaries give a π phase shift, which matches that of a classical wave reflected at a hard boundary; whether there is a deeper connection I'm not sure!
[1 mark] for something sensible.
In any case, the soft turning points lead to the extra factor of 1/2, as there are two soft boundaries per cycle.
[1 mark]

B.2 Using the Bohr Sommerfeld quantization condition, find the WKB approximation for energy of the quantum particle in the harmonic oscillator.

[6 marks]

$$p(x) = \sqrt{2mE - m^2\omega^2 x^2}$$

[1 mark]
where the $+$ sign is chosen by convention. In the Bohr Sommerfeld condition, the integral is twice the distance between the classical turning points.
[1 mark]
Therefore

$$\begin{aligned}\oint p(x)\,\mathrm{d}x &= 2\int_{x_-}^{x_+} \sqrt{2mE - m^2\omega^2 x^2}\mathrm{d}x \\ &= 2\sqrt{2mE}\int_{x_-}^{x_+} \sqrt{1 - \frac{m\omega^2}{2E}x^2}\mathrm{d}x\end{aligned}$$

now change variables using

$$\begin{aligned}\sin(\theta) &= \sqrt{\frac{m\omega^2}{2E}}x \\ \cos(\theta)\,\mathrm{d}\theta &= \sqrt{\frac{m\omega^2}{2E}}\mathrm{d}x\end{aligned}$$

[1 mark]
and the integration limits are now

$$\begin{aligned}x_\pm &= \pm\sqrt{\frac{2E}{m\omega^2}} \\ &\downarrow \\ \sin(\theta) &= \pm 1 \\ \theta &= \pm\pi/2\end{aligned}$$

[1 mark]

$$\begin{aligned}\oint p(x)\,\mathrm{d}x &= 2\sqrt{2mE}\sqrt{\frac{2E}{m\omega^2}}\int_{-\pi/2}^{\pi/2}\cos^2(\theta)\,\mathrm{d}\theta \\ &= \frac{2E}{\omega}\int_{-\pi/2}^{\pi/2}(1+\cos(2\theta))\,\mathrm{d}\theta \\ &= \frac{2E}{\omega}\pi\end{aligned}$$

[1 mark]
Therefore

$$\frac{2E}{\omega}\pi = 2\pi\hbar\left(n + \frac{1}{2}\right)$$

and

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right).$$

[1 mark]

B.3 Explain for which energies the WKB approximation to the wavefunction will be the most accurate.

[2 marks]

The WKB approximation is semi-classical, and the correspondence principle states that quantum mechanics approaches classical mechanics in the limit of large quantum number. Hence, it is most accurate for large energies. Any statement to a similar effect receives full marks.

Section C: more challenging.

C.1 Explain with the aid of a diagram the possible trajectories undertaken by the quantum particle through phase space (x, p), including the areas enclosed.

[5 marks]

$$\hbar\omega\left(n+\frac{1}{2}\right)=\frac{p\left(x\right)^{2}}{2m}+\frac{1}{2}m\omega^{2}x^{2}$$

This set of equations describes concentric ellipses centred on $(p, x) = \mathbf{0}$.
[1 mark]
It will be easiest to plot using scaled axes:

$$n+\frac{1}{2}=p'^{2}+x'^{2}$$

$$p' = p/\sqrt{2m\hbar\omega}$$
$$x' = x\sqrt{\frac{m\omega}{2\hbar}}$$

in which case the classical trajectories are simply concentric circles of radius $\sqrt{n+\frac{1}{2}}$.
[1 mark]
The Bohr Sommerfeld condition tells us that the area enclosed in (x, p) by ellipse n must be $(n + 1/2)\,\hbar\omega$, or area $\pi\left(n+\frac{1}{2}\right)$ in (x', p') which happens to be equal to the full quantum solution for the harmonic oscillator.
[1 mark]
Finally,
[2 marks]
For a decent picture summarising the above. Rescaling is not important provided the general shapes are correct, or axis intercepts are clearly labelled.

3 Semiclassics (assorted questions not in exam style)

3.1 Airy function

Show that the Airy function, defined by

$$\mathrm{Ai}\left(x\right) = \frac{1}{\pi}\int_0^\infty \mathrm{d}t \cos\left(\frac{t^3}{3} + xt\right) \tag{14}$$

solves the Airy equation

$$y'' - xy = 0. \tag{15}$$

Hint: find y'' by differentiating under the integral. Then subtract xy from the result, and notice the the resulting integral can be done by inspection. You need the fact that

$$\sin\left(\frac{t^3}{3} + xt\right)\Big|_{t\to\infty} = 0. \tag{16}$$

NB this approach is a little imprecise; the fully correct way employs contour integration.

[4 marks]

3.2 Gaussian integrals and the action

The simple harmonic oscillator (as usual) turns out to be exactly solvable at the propagator level. In this question we will see the solution in full detail, which will also clarify various bits I glossed over in lectures.

3.2.1 Euclidean action

In the lectures we saw how to do functional Gaussian integrals of the form:

$$\int \mathcal{D}x \exp\left(-\frac{1}{2}\int_0^t x\left(t'\right)\hat{A}x\left(t'\right)\mathrm{d}t'\right) = \sqrt{\frac{\left(2\pi\right)^\infty}{\det\left(\hat{A}\right)}}. \tag{17}$$

Show that the usual harmonic oscillator action

$$iS\left[x\right] = i\int\left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2x^2\right)\mathrm{d}t \tag{18}$$

transforms to the Euclidean action

$$-S_E\left[x\right] = \int\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega^2x^2\right)\mathrm{d}\tau \tag{19}$$

under Wick rotation $t \to -i\tau$.

[3 marks]

3.2.2 Gaussian form

Using integration by parts, show that

$$\int \mathcal{D}x \exp\left(-S_E\left[x\right]/\hbar\right) = \int \mathcal{D}x \exp\left(-\frac{1}{2}\int x\left(\tau\right)\hat{A}x\left(\tau\right)\mathrm{d}\tau/\hbar\right) \tag{20}$$

with

$$\hat{A} = m\left(\frac{\partial^2}{\partial\tau^2} - \omega^2\right). \tag{21}$$

[3 marks]

3.2.3 Wick rotate back

By Wick rotating back, show that the real-time propagator for the simple harmonic oscillator is

$$K_{\mathrm{SHO}}\left(x=0,t;x=0,t=0\right) \triangleq \int \mathcal{D}x \exp\left(i\int_0^t x\left(t'\right)\hat{A}x\left(t'\right)\mathrm{d}t'/\hbar\right) \tag{22}$$

$$= \sqrt{\frac{\left(2\pi i\right)^{\infty}}{\det\left(\hat{A}\right)}} \tag{23}$$

with

$$\hat{A} = -\frac{m}{2}\left(\frac{\partial^2}{\partial t'^2}+\omega^2\right). \tag{24}$$

[2 marks]

(NB The Wick rotation is needed to get convergent integrals along the way, but the result is the same as if we'd ignored the divergence).

3.2.4 Calculating the pre-factor: the determinant.

We'll generally neglect the mysterious pre-factor of the Gaussian functional integral. But let's calculate it once, here, in this exactly solvable case. See Altland and Simons, *Condensed Matter Field Theory*, for more details. We require

$$\det\left(\hat{A}\right) = \det\left(-\frac{m}{2}\left(\frac{\partial^2}{\partial t'^2}+\omega^2\right)\right). \tag{25}$$

The trick is to recall that the determinant of an operator is just the product of its eigenvalues. These are found by solving

$$-\frac{m}{2}\left(\frac{\partial^2}{\partial t'^2}+\omega^2\right)x_n\left(t'\right) = \epsilon_n x_n\left(t'\right) \tag{26}$$

subject to the boundary conditions

$$x_n\left(t\right) = x_n\left(0\right) = 0. \tag{27}$$

Show that

$$x_n\left(t'\right) = \sin\left(\frac{n\pi t'}{t}\right) \tag{28}$$

are such eigenfunctions, with

$$\epsilon_n = \frac{m}{2}\left(\left(\frac{n\pi}{t}\right)^2-\omega^2\right) \tag{29}$$

the corresponding eigenvalues.

[4 marks]

3.2.5 Almost there...

Hence show that

$$K_{\mathrm{SHO}}\left(x=0,t;x=0,t=0\right) = \left(2\pi i\right)^{\infty}\prod_{n=1}^{\infty}\left(\frac{m}{2}\left(\left(\frac{n\pi}{t}\right)^2-\omega^2\right)\right)^{-1/2}.$$

[2 marks]

3.2.6 Cancelling the infinity

The trick to dealing with the horrendous infinity out the front is to only deal with quantities that have it cancel. Note that when the potential $V \to 0$, $K_{\mathrm{SHO}} = K_{\mathrm{free}}$, we should obtain the propagator for the free particle:

$$K_{\mathrm{free}}\left(0,t;0,0\right) = \sqrt{\frac{m}{2\pi i\hbar t}}. \tag{30}$$

Hence show that

$$K_{\mathrm{SHO}}\left(0,t;0,0\right) = \sqrt{\frac{m}{2\pi i\hbar t}}\prod_{n}\left(1-\left(\frac{\omega t}{n\pi}\right)^{2}\right)^{-1/2}.$$

[4 marks]

3.2.7 Final answer...

Use the fact that

$$\prod_{n=1}^{\infty}\left(1-\left(\frac{x}{n\pi}\right)^{2}\right)^{-1} = \frac{1}{\mathrm{sinc}\left(x\right)} \tag{31}$$

To find the propagator for the SHO.

[2 marks]

$$K_{\mathrm{SHO}} = \sqrt{\frac{m\omega}{2\pi i\hbar\sin\left(\omega t\right)}}$$