Advanced Quantum Physics: Problem Set 2

1 Path Integral Quantum Mechanics

Section A: mostly bookwork.

A.1 The time evolution operator $\hat{U}(t - t_0)$ is defined by

$$|\psi(t)\rangle = \hat{U}(t - t_0)\,|\psi(t_0)\rangle. \tag{1}$$

Show that for probability to be conserved, $\hat{U}$ must be unitary.

[4 marks]

A.2 A particle is initially at position x_0 at time t_0. Explain why the amplitude to find the particle at position x at time T is given by the propagator

$$K(x, T; x_0, t_0)\,. \tag{2}$$

[6 marks]

Section B: bringing together ideas from across the course.

B.1 A particle starts at position $x = 0$ at time $t = 0$. At time t' it travels through a slit of width $2b$ centred about $x = 0$ before reaching a screen at time T. Explain with the aid of a diagram why the amplitude to find the particle at position x on the screen is given by

$$\psi(x, T) = \int_{-b}^{b} \mathrm{d}x' K(x, T; x', t')\, K(x', t'; 0, 0)\,. \tag{3}$$

[4 marks]

B.2 The propagator in free space is given by

$$K(x_f, t_f; x_i, t_i) = \sqrt{\frac{m}{2\pi\hbar i}}\,\frac{1}{\sqrt{t_f - t_i}}\exp\left(i\frac{m(x_f - x_i)^2}{2\hbar(t_f - t_i)}\right).$$

Show that the amplitude to find the particle in **B.1** at position x at time T is given by

$$\psi(x, T) = \frac{1}{2\pi i}\exp\left(i\frac{mx^2}{2\hbar T}\right)\sqrt{\frac{2m}{\hbar T}}\int_{-a(x)}^{a(x)} \mathrm{d}y \exp\left(iy^2\right)$$

for a suitably defined $a(x)$.

[6 marks]

Section C: more challenging.

The *Fresnel integrals* are defined as

$$C(x) \triangleq \int_0^x \mathrm{d}y \cos\left(y^2\right)$$
$$S(x) \triangleq \int_0^x \mathrm{d}y \sin\left(y^2\right).$$

C.1 Show that the intensity at position x on the screen can be written as

$$I(x,T) = \frac{2m}{\hbar\pi^2 T}\left(C^2(a) + S^2(a)\right).$$

Hint: You will need to use the fact that both $C(x)$ and $S(x)$ are odd functions.

[5 marks]

Path Integral Quantum Mechanics: Solutions

Section A: mostly bookwork.

A.1 The time evolution operator $\hat{U}\left(t-t_0\right)$ is defined by

$$|\psi\left(t\right)\rangle = \hat{U}\left(t-t_0\right)|\psi\left(t_0\right)\rangle. \quad (4)$$

Show that for probability to be conserved, $\hat{U}$ must be unitary.

[4 marks]

Probability is the square modulus of the amplitude (the Born rule).
[1 mark]
Therefore we require

$$\left|\langle\psi\left(t\right)|\psi\left(t\right)\rangle\right|^2 = \left|\langle\psi\left(t_0\right)|\psi\left(t_0\right)\rangle\right|^2$$

[1 mark]
and

$$\langle\psi\left(t\right)|\psi\left(t\right)\rangle = \exp\left(i\phi\right)\langle\psi\left(t_0\right)|\psi\left(t_0\right)\rangle$$

where ϕ is some real phase. However, we know what this phase is, since a proper normalisation of a state is defined as

$$\langle\psi|\psi\rangle = 1$$

and so $\phi = 0$. If the phase isn't commented on, that's fine!
From the stated equation,

$$\langle\psi\left(t\right)|\psi\left(t\right)\rangle = \langle\psi\left(t_0\right)|\hat{U}^\dagger\hat{U}|\psi\left(t_0\right)\rangle$$

[1 mark].
Therefore we require

$$\hat{U}^\dagger\hat{U} = \hat{\mathbb{I}}$$

which is the definition of unitarity.
[1 mark]

A.2 A particle is initially at position x_0 at time t_0. Explain why the amplitude to find the particle at position x at time T is given by the propagator

$$K\left(x,T;x_0,t_0\right). \quad (5)$$

[6 marks]

This is a long derivation, but it's from the notes. First project the equation given in A1 to the position basis:

$$\langle x|\psi\left(T\right)\rangle = \langle x|\hat{U}\left(T-t_0\right)|\psi\left(t_0\right)\rangle$$

[1 mark].
Now insert a decomposition of the identity into the position basis:

$$\hat{\mathbb{I}} = \int \mathrm{d}x' |x'\rangle\langle x'|$$

[1 mark]
to give

$$\langle x|\psi(T)\rangle = \int \mathrm{d}x' \langle x|\hat{U}(T-t_0)|x'\rangle\langle x'|\psi(t_0)\rangle$$

[1 mark].
The propagator is defined to be

$$K(x,T;x',t_0) = \langle x|\hat{U}(T-t_0)|x'\rangle$$

and so we have

$$\langle x|\psi(T)\rangle = \int \mathrm{d}x' K(x,T;x',t_0)\langle x'|\psi(t_0)\rangle$$

[1 mark]
Finally, since the particle was stated to be initially at a definite position x_0, it must have been described at the Dirac delta function at that instant:

$$\langle x'|\psi(t_0)\rangle = \delta(x'-x_0)$$

[1 mark]
giving

$$\begin{aligned}\langle x|\psi(T)\rangle &= \int \mathrm{d}x' K(x,T;x',t_0)\,\delta(x'-x_0)\\ \psi(x,T) &= K(x,T;x_0,t_0)\end{aligned}$$

as required.
[1 mark]

Section B: bringing together ideas from across the course.

B.1 In Fig. 1 A particle starts at position $x=0$ at time $t=0$. At time t' it travels through a slit of width $2b$ centred about $x=0$ before reaching a screen at time T. Explain with the aid of a diagram why the amplitude to find the particle at position x on the screen is given by

$$\psi(x,T) = \int_{-b}^{b} \mathrm{d}x' K(x,T;x',t')\,K(x',t';0,0)\,. \tag{6}$$

[4 marks]

In quantum mechanics amplitudes play the role of probabilities in classical problems. In particular,

$$\begin{aligned}\mathrm{Amp}(A \text{ and } B) &= \mathrm{Amp}(A)\,\mathrm{Amp}(B)\\ \mathrm{Amp}(A \text{ or } B) &= \mathrm{Amp}(A) + \mathrm{Amp}(B)\,.\end{aligned}$$

For the particle to reach point x on the screen, it must first reach point x' within the slit: this is a case of 'A and B', with $A=$(particle reaches x,T) and $B=$(particle reaches x',t'). But this is true for all allowed positions x' within the slit, so we must sum over these possibilities. This is a case of 'A or B' (or C or D...), with A, B, etc labelling all the points within the slit.
[2 marks] for any reasonable explanation

[2 marks] for a decent picture with relevant points labelled.

B.2 The propagator in free space is given by

$$K\left(x_f, t_f; x_i, t_i\right) = \sqrt{\frac{m}{2\pi\hbar i}} \frac{1}{\sqrt{t_f - t_i}} \exp\left(i \frac{m\left(x_f - x_i\right)^2}{2\hbar\left(t_f - t_i\right)}\right).$$

Show that the amplitude to find the particle in **B.1**. at position x at time T is given by

$$\psi\left(x, T\right) = \frac{1}{2\pi i} \exp\left(i \frac{m x^2}{2\hbar T}\right) \sqrt{\frac{2m}{\hbar T}} \int_{-a(x)}^{a(x)} \mathrm{d}y \exp\left(i y^2\right)$$

for a suitably defined $a\left(x\right)$.

[6 marks]

I'll be honest here: the algebra required for this question is much harder than anything you would realistically be required to perform in an exam without a lot more help. I wanted to give you exam-style questions so you have some idea what to expect, but I also wanted to cover relevant material thoroughly. So let's give it a go, but don't worry if it looks like too much – it is!

$$\begin{aligned}
\psi\left(x, T\right) &= \int_{-b}^{b} \mathrm{d}x' K\left(x, T; x', t'\right) K\left(x', t'; 0, 0\right) \\
&= \frac{m}{2\pi\hbar i} \frac{1}{\sqrt{t'\left(T - t'\right)}} \int_{-b}^{b} \mathrm{d}x' \exp\left(i \frac{m}{2\hbar}\left(\frac{\left(x - x'\right)^2}{\left(T - t'\right)} + \frac{x'^2}{t'}\right)\right)
\end{aligned}$$

[1 mark]

$$\begin{aligned}
\psi\left(x, T\right) &= \frac{m}{2\pi\hbar i} \frac{1}{\sqrt{t'\left(T - t'\right)}} \int_{-b}^{b} \mathrm{d}x' \exp\left(i \frac{m}{2\hbar\left(T - t'\right)}\left(\left(x - x'\right)^2 + x'^2 \frac{T - t'}{t'}\right)\right) \\
&= \frac{m}{2\pi\hbar i} \frac{1}{\sqrt{t'\left(T - t'\right)}} \int_{-b}^{b} \mathrm{d}x' \exp\left(i \frac{m}{2\hbar\left(T - t'\right)}\left(x^2 + x'^2 - 2xx' + x'^2\left(\frac{T}{t'} - 1\right)\right)\right) \\
&= \frac{m}{2\pi\hbar i} \frac{1}{\sqrt{t'\left(T - t'\right)}} \int_{-b}^{b} \mathrm{d}x' \exp\left(i \frac{m}{2\hbar\left(T - t'\right)}\left(\frac{T}{t'}\right)\left(x'^2 - \left(\frac{t'}{T}\right)2xx' + \left(\frac{t'}{T}\right)x^2\right)\right)
\end{aligned}$$

Now complete the square:

$$\psi\left(x, T\right) = \frac{m}{2\pi\hbar i} \frac{1}{\sqrt{t'\left(T - t'\right)}} \int_{-b}^{b} \mathrm{d}x' \exp\left(i \frac{m}{2\hbar\left(T - t'\right)}\left(\frac{T}{t'}\right)\left(\left(x' - \left(\frac{t'}{T}\right)x\right)^2 + \frac{t'}{T}\left(1 - \frac{t'}{T}\right)x^2\right)\right)$$

[2 marks]
The x^2 term is not a function of x' so pulls out of the intergral:

$$\psi\left(x, T\right) = \frac{m}{2\pi\hbar i} \frac{1}{\sqrt{t'\left(T - t'\right)}} \exp\left(i \frac{m x^2}{2\hbar T}\right) \int_{-b}^{b} \mathrm{d}x' \exp\left(i \frac{m}{2\hbar\left(T - t'\right)} \frac{T}{t'}\left(x' - \left(\frac{t'}{T}\right)x\right)^2\right)$$

[1 mark]
Now change variables:

$$y' = \sqrt{\frac{m}{2\hbar\left(T - t'\right)} \frac{T}{t'}} x'$$

and define

$$b' = \sqrt{\frac{m}{2\hbar\left(T-t'\right)}\frac{T}{t'}}b$$

giving

$$\psi\left(x,T\right) = \frac{1}{2\pi i}\exp\left(i\frac{mx^2}{2\hbar T}\right)\sqrt{\frac{2m}{\hbar T}}\int_{-b'}^{b'}\mathrm{d}y'\exp\left(i\left(y'-\sqrt{\frac{m}{2\hbar\left(T-t'\right)}\frac{T}{t'}}\left(\frac{t'}{T}\right)x\right)^2\right)$$

And change variables again to remove the x term in the integrand:

$$y = y' - \sqrt{\frac{m}{2\hbar\left(T-t'\right)}\frac{T}{t'}}\left(\frac{t'}{T}\right)x$$

[1 mark]
defining

$$\begin{aligned} a\left(x\right) &= b' - \sqrt{\frac{m}{2\hbar\left(T-t'\right)}\frac{T}{t'}}\left(\frac{t'}{T}\right)x \\ &= \sqrt{\frac{m}{2\hbar\left(T-t'\right)}\frac{T}{t'}}\left(b-\left(\frac{t'}{T}\right)x\right) \end{aligned}$$

[1 mark]
gives

$$\psi\left(x,T\right) = \frac{1}{2\pi i}\exp\left(i\frac{mx^2}{2\hbar T}\right)\sqrt{\frac{2m}{\hbar T}}\int_{-a}^{a}\mathrm{d}y\exp\left(iy^2\right).$$

[I wouldn't be surprised if I've made some errors in my own algebra here; please let me know if you disagree. As a basic check, ψ does at least have the correct units.]

Section C: more challenging.

The *Fresnel integrals* are defined as

$$\begin{aligned} C\left(x\right) &\triangleq \int_0^x \mathrm{d}y\cos\left(y^2\right) \\ S\left(x\right) &\triangleq \int_0^x \mathrm{d}y\sin\left(y^2\right). \end{aligned}$$

C.1 Show that the intensity at position x on the screen can be written as

$$I\left(x,T\right) = \frac{2m}{\hbar\pi^2 T}\left(C^2\left(a\right)+S^2\left(a\right)\right).$$

Hint: You will need to use the fact that both $C\left(x\right)$ and $S\left(x\right)$ are odd functions.

[5 marks]

$$\begin{aligned}
\int_{-a}^{a} \mathrm{d}y \exp\left(iy^2\right) &= \int_{0}^{a} \mathrm{d}y \exp\left(iy^2\right) + \int_{-a}^{0} \mathrm{d}y \exp\left(iy^2\right) \\
&= \int_{0}^{a} \mathrm{d}y \exp\left(iy^2\right) - \int_{0}^{-a} \mathrm{d}y \exp\left(iy^2\right) \\
&= C(a) + iS(a) - C(-a) - iS(-a) \\
&= 2\left(C(a) + iS(a)\right)
\end{aligned}$$

where the last line used that the functions are both odd.
[2 marks]
Therefore

$$\psi(x,T) = \frac{1}{\pi i} \exp\left(i\frac{mx^2}{2\hbar T}\right)\sqrt{\frac{2m}{\hbar T}}\left(C(a) + iS(a)\right).$$

[1 mark]

$$I(x,T) = \left|\psi(x,T)\right|^2$$

[1 mark]
and so

$$I(x,T) = \frac{2m}{\hbar \pi^2 T}\left(C^2(a) + S^2(a)\right).$$

[1 mark]

2 Path Integral Quantum Mechanics

Section A: mostly bookwork.

In this question you will calculate the amplitude for a quantum particle at an initial positon x_i at time t_i to move to position x_f at time t_f, in the special case $V = 0$.

A.1 Explain the meaning of the following terms, giving mathematical expressions in each case.

(i) The time evolution operator $\hat{U}\left(t_f - t_i\right)$.

[3 marks]

(ii) The propagator $K\left(x_f, t_f; x_i, t_i\right)$.

[3 marks]

A.2 For a particle in free space, $V = 0$, explain why

$$K\left(x_f, t_f; x_i, t_i\right) = \langle x_f| \exp\left(-i\hat{p}^2\left(t_f - t_i\right)/2m\hbar\right)|x_i\rangle \tag{7}$$

where $\hat{p}$ is the momentum operator and m is the mass of the particle.

[4 marks]

Section B: bringing together ideas from across the course.

B.1 Define the Gaussian integral

$$I\left(a\right) = \int_{-\infty}^{\infty} \exp\left(-ax^2\right) \mathrm{d}x. \tag{8}$$

By considering I^2, and working in plane polar co-ordinates, show that

$$I\left(a\right) = \sqrt{\frac{\pi}{a}}. \tag{9}$$

[5 marks]

B.2 Now consider the Gaussian integral

$$I\left(a, b\right) = \int_{-\infty}^{\infty} \exp\left(-ax^2 + bx\right) \mathrm{d}x. \tag{10}$$

By completing the square, or otherwise, show that

$$I\left(a, b\right) = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a}\right). \tag{11}$$

[3 marks]

B.3 By resolving the identity operator in the momentum basis, or otherwise, show that

$$\exp\left(-i\hat{p}^2\left(t_f - t_i\right)/2m\right) = \int_{-\infty}^{\infty} \mathrm{d}p \exp\left(-ip^2\left(t_f - t_i\right)/2m\right)|p\rangle\langle p| \tag{12}$$

where

$$\hat{p}|p\rangle = p|p\rangle. \tag{13}$$

[2 marks]

Section C: more challenging.

C.1 Show that the propagator in free space is given by

$$K\left(x_f, t_f; x_i, t_i\right) = \sqrt{\frac{m}{2\pi i\hbar}}\frac{1}{\sqrt{t_f - t_i}}\exp\left(\frac{i\left(x_f - x_i\right)^2 m}{2\hbar\left(t_f - t_i\right)}\right). \tag{14}$$

Hint: you will need to use the answers to A2, B2, and B3. You may use the fact that

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp\left(ipx\right). \tag{15}$$

[5 marks]

Answers to Question 2

Section A: mostly bookwork.

In this question you will calculate the amplitude for a quantum particle at an initial positon x_i at time t_i to move to position x_f at time t_f, in the special case $V = 0$.

A.1 Explain the meaning of the following terms, giving mathematical expressions in each case.

(i) The time evolution operator $\hat{U}\left(t_f - t_i\right)$

[3 marks]

The time evolution operator acting on a state $|\psi\left(t_i\right)\rangle$ takes it to state $|\psi\left(t_f\right)\rangle$.
[1 mark]
That is,

$$\hat{U}\left(t_f - t_i\right)|\psi\left(t_i\right)\rangle = |\psi\left(t_f\right)\rangle \tag{16}$$

[1 mark].
Explicitly, it is given by

$$\hat{U}\left(t_f - t_i\right) = \exp\left(-i\hat{H}\left(t_f - t_i\right)/\hbar\right) \tag{17}$$

[1 mark]. There will be 3 marks for any 3 relevant comments.

(ii) The propagator $K\left(x_f, t_f; x_i, t_i\right)$.

[3 marks]

The propagator is the amplitude to find a state $|x_f\left(t_f\right)\rangle$ given an initial state $|x_i\left(t_i\right)\rangle$.
[1 mark].
That is,

$$K\left(x_f, t_f; x_i, t_i\right) = \langle x_f\left(t_f\right)|\hat{U}\left(t_f - t_i\right)|x_i\left(t_i\right)\rangle. \tag{18}$$

[2 marks].

A.2 For a particle in free space, $V = 0$, explain why

$$K\left(x_f, t_f; x_i, t_i\right) = \langle x_f|\exp\left(-i\hat{p}^2\left(t_f - t_i\right)/2m\hbar\right)|x_i\rangle \tag{19}$$

where $\hat{p}$ is the momentum operator and m is the mass of the particle.

[4 marks]

In free space, the potential is zero.
[1 mark]
Therefore

$$\hat{H} = \hat{T} = \hat{p}^2/2m$$

[1 mark]
Since

$$K\left(x_f,t_f;x_i,t_i\right)=\langle x_f|\hat{U}\left(t_f-t_i\right)|x_i\rangle \tag{20}$$
$$=\langle x_f|\exp\left(-i\hat{H}\left(t_f-t_i\right)/\hbar\right)|x_i\rangle \tag{21}$$

[2 marks]
this gives

$$K\left(x_f,t_f;x_i,t_i\right)=\langle x_f|\exp\left(-i\hat{p}^2\left(t_f-t_i\right)/2m\hbar\right)|x_i\rangle. \tag{22}$$

In the part worth 2 marks I would be generous if the same expression were not noted in part (ii).

Section B: bringing together ideas from across the course.

B.1 Define the Gaussian integral

$$I\left(a\right)=\int_{-\infty}^{\infty}\exp\left(-ax^2\right)\mathrm{d}x. \tag{23}$$

By considering I^2, and working in plane polar co-ordinates, show that

$$I\left(a\right)=\sqrt{\frac{\pi}{a}}. \tag{24}$$

[5 marks]

$$I^2=\left(\int_{-\infty}^{\infty}\exp\left(-ax^2\right)\mathrm{d}x\right)^2 \tag{25}$$
$$=\left(\int_{-\infty}^{\infty}\exp\left(-ax^2\right)\mathrm{d}x\right)\left(\int_{-\infty}^{\infty}\exp\left(-ay^2\right)\mathrm{d}y\right) \tag{26}$$

(using a different label for the dummy integral in the second case). Therefore

$$I^2=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-a\left(x^2+y^2\right)\right)\mathrm{d}x\mathrm{d}y \tag{27}$$

[1 mark]
Now switch to plane polar co-ordinates. You can derive the Jacobian, or just remember it (dimensions pretty much fix what it can be!):

$$I^2=\int_0^{2\pi}\mathrm{d}\theta\int_0^{\infty}\mathrm{d}r\cdot r\exp\left(-ar^2\right) \tag{28}$$

[1 mark].
The θ integral separates out:

$$I^2=2\pi\int_0^{\infty}\mathrm{d}r\cdot r\exp\left(-ar^2\right) \tag{29}$$

[1 mark]
and the remaining integral can be done by inspection:

$$I^2 = 2\pi \left[\frac{-1}{2a} \exp\left(-ar^2\right) \right]_0^\infty \tag{30}$$

[1 mark]
giving

$$I^2 = \frac{\pi}{a} \tag{31}$$

[1 mark]
and

$$I = \sqrt{\frac{\pi}{a}}. \tag{32}$$

B.2 Now consider the Gaussian integral

$$I(a,b) = \int_{-\infty}^{\infty} \exp\left(-ax^2 + bx\right) \mathrm{d}x. \tag{33}$$

By completing the square, or otherwise, show that

$$I(a,b) = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a}\right). \tag{34}$$

[3 marks]

$$\begin{aligned} I(a,b) &= \int_{-\infty}^{\infty} \exp\left(-ax^2 + bx\right) \mathrm{d}x && (35) \\ &= \int_{-\infty}^{\infty} \exp\left(-a\left(x^2 - \frac{bx}{a}\right)\right) \mathrm{d}x && (36) \\ &= \int_{-\infty}^{\infty} \exp\left(-a\left(x - \frac{b}{2a}\right)^2 + a\left(\frac{b}{2a}\right)^2\right) \mathrm{d}x && (37) \end{aligned}$$

[1 mark].
The final term is not a function of x, so it factors out:

$$I(a,b) = \exp\left(\frac{b^2}{4a}\right) \int_{-\infty}^{\infty} \exp\left(-a\left(x - \frac{b}{2a}\right)^2\right) \mathrm{d}x. \tag{38}$$

Finally, use a change of variables

$$y = x - \frac{b}{2a} \tag{39}$$

[1 mark].
Since this is a finite shift, neither infinite limite is affected, and the original Gaussian integral results:

$$I(a,b) = \exp\left(\frac{b^2}{4a}\right) \int_{-\infty}^{\infty} \exp\left(-ay^2\right) \mathrm{d}y. \tag{40}$$

[1 mark]
Hence,

$$I\left(a,b\right)=\sqrt{\frac{\pi}{a}}\exp\left(\frac{b^2}{4a}\right).$$

B.3 By resolving the identity operator in the momentum basis, or otherwise, show that

$$\exp\left(-i\hat{p}^2\left(t_f-t_i\right)/2m\right)=\int_{-\infty}^{\infty}\mathrm{d}p\exp\left(-ip^2\left(t_f-t_i\right)/2m\right)|p\rangle\langle p| \quad (41)$$

where

$$\hat{p}|p\rangle=p|p\rangle. \quad (42)$$

[2 marks]

Resolving the identity operator in the momentum basis means

$$\hat{\mathbb{I}}=\int_{-\infty}^{\infty}\mathrm{d}p|p\rangle\langle p| \quad (43)$$

[1 mark].
Stick it on the right, say:

$$\exp\left(-i\hat{p}^2\left(t_f-t_i\right)/2m\right)\int_{-\infty}^{\infty}\mathrm{d}p|p\rangle\langle p|. \quad (44)$$

Now, the slightly strange bit is that $\hat{p}$ the operator is not a function of p the eigenstate(!). But if you think of matrices passing through sums over their eigenstates perhaps that gives some intuition. The result is

$$\int_{-\infty}^{\infty}\mathrm{d}p\exp\left(-i\hat{p}^2\left(t_f-t_i\right)/2m\right)|p\rangle\langle p| \quad (45)$$

$$=\int_{-\infty}^{\infty}\mathrm{d}p\exp\left(-ip^2\left(t_f-t_i\right)/2m\right)|p\rangle\langle p| \quad (46)$$

[1 mark]
where the hat has disappeared because

$$\hat{p}|p\rangle=p|p\rangle \quad (47)$$

and the exponential of the function of $\hat{p}$ is just defined by its Taylor series.

Section C: more challenging.

C.1 Show that the propagator in free space is given by

$$K\left(x_f,t_f;x_i,t_i\right)=\sqrt{\frac{m}{2\pi i\hbar}}\frac{1}{\sqrt{t_f-t_i}}\exp\left(\frac{i\left(x_f-x_i\right)^2m}{2\hbar\left(t_f-t_i\right)}\right). \quad (48)$$

Hint: you will need to use the answers to A2, B2, and B3. You may use the fact that

$$\langle x|p\rangle=\frac{1}{\sqrt{2\pi\hbar}}\exp\left(ipx\right). \quad (49)$$

[5 marks]

OK, so stick it all together! In A2 we have

$$K\left(x_f,t_f;x_i,t_i\right)=\langle x_f|\exp\left(-i\hat{p}^2\left(t_f-t_i\right)/2m\hbar\right)|x_i\rangle. \tag{50}$$

Insert a complete set of momentum states as in B2:

$$K\left(x_f,t_f;x_i,t_i\right)=\int_{-\infty}^{\infty}\mathrm{d}p\langle x_f|\exp\left(-i\hat{p}^2\left(t_f-t_i\right)/2m\hbar\right)|p\rangle\langle p|x_i\rangle. \tag{51}$$

[1 mark].
Therefore the $\hat{p}$ loses its hat, as in B3:

$$K\left(x_f,t_f;x_i,t_i\right)=\int_{-\infty}^{\infty}\mathrm{d}p\langle x_f|\exp\left(-ip^2\left(t_f-t_i\right)/2m\hbar\right)|p\rangle\langle p|x_i\rangle. \tag{52}$$

[1 mark]
Now the exponential is just a complex number, not an operator, so it pulls out to the left:

$$K\left(x_f,t_f;x_i,t_i\right)=\int_{-\infty}^{\infty}\mathrm{d}p\exp\left(-ip^2\left(t_f-t_i\right)/2m\hbar\right)\langle x_f|p\rangle\langle p|x_i\rangle. \tag{53}$$

[1 mark]
We were reminded of the Dirac notation for a plane wave in the Hint, which tells us that

$$K\left(x_f,t_f;x_i,t_i\right)=\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\mathrm{d}p\exp\left(-ip^2\left(t_f-t_i\right)/2m\hbar\right)\exp\left(ipx_f/\hbar\right)\exp\left(-ipx_i/\hbar\right) \tag{54}$$

and so

$$K\left(x_f,t_f;x_i,t_i\right)=\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\mathrm{d}p\exp\left(-ip^2\left(t_f-t_i\right)/2m\hbar+ip\left(x_f-x_i\right)/\hbar\right). \tag{55}$$

[1 mark]
Now notice that this is just a Gaussian integral of the form in B2:

$$\int_{-\infty}^{\infty}\mathrm{d}x\exp\left(-ax^2+bx\right)=\sqrt{\frac{\pi}{a}}\exp\left(\frac{b^2}{4a}\right) \tag{56}$$

with

$$\begin{aligned}a&=i\left(t_f-t_i\right)/2m\hbar\\ b&=i\left(x_f-x_i\right)/\hbar.\end{aligned}$$

[1 mark]
If you're worried about the i, that's good! But in fact it turns out the Gaussian integral can be done for complex exponents without any issue provided the real part is negative. A comment on this (even to say you're unsure) would be welcome but is not necessary.
Therefore

$$K\left(x_f,t_f;x_i,t_i\right)=\sqrt{\frac{m}{2\pi i\left(t_f-t_i\right)\hbar}}\exp\left(\frac{im\left(x_f-x_i\right)^2}{2\left(t_f-t_i\right)\hbar}\right). \tag{57}$$