Advanced Quantum Physics: Problem Set 1

1 Lagrangian Mechanics

Section A: mostly bookwork.

Using generalised co-ordinates $q_i(t)$ and velocities $\dot{q}_i(t)$ at time t the Lagrangian for a particle moving in a potential $V(q)$ is

$$L(q_i, \dot{q}_i) = \frac{1}{2}m\dot{q}^2 - V(q)\,. \tag{1}$$

A.1 Write down the corresponding action governing the motion between times t_0 and t_1.

[2 marks]

A.2 Call the true trajectory of the particle $q_i(t)$. By considering variations away from this trajectory of the form $q_i(t) + \lambda\epsilon_i(t)$, formulate a mathematical statement of the principle of least action, explaining your reasoning.

[3 marks]

A.3 Using the principle of least action, derive the Euler Lagrange equations describing the motion of the particle.

[5 marks]

Section B: bringing together ideas from across the course.

Now consider a simple example of functional calculus in a different context.

The distance between two points is given by the functional

$$l[\boldsymbol{x}] = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \tag{2}$$

where

$$\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}. \tag{3}$$

B.1 Explain the origin of Eq. 2.

[3 marks]

B.2 By considering paths $\boldsymbol{x} + \lambda\boldsymbol{\epsilon}$, find a differential equation describing the path minimising the distance between x_1 and x_2. Explain all the steps in your working.

[4 marks]

B.3 Solve your equation to find the shortest distance between the two points. Does this match your intuition?

[3 marks]

Section C: more challenging.

C.1 The squared proper distance between two spacetime events in Minkowski space is

$$\mathrm{d}s^2 = \mathrm{d}x^2 - c^2\mathrm{d}t^2. \tag{4}$$

Find the path $t\left(x\right)$ that minimises the proper distance between two spacetime events.
[Hint: you may use your answers to Section B.]

[3 marks]

C.2 What is the proper distance along this path?

[2 marks]

Solutions to Question 1

Section A: mostly bookwork.

Using generalised co-ordinates $q_i(t)$ and velocities $\dot{q}_i(t)$ at time t the Lagrangian for a particle moving in a potential $V(q)$ is

$$L(q_i, \dot{q}_i, t) = \frac{1}{2} m \dot{q}^2 - V(q). \tag{5}$$

A.1 Write down the corresponding action governing the motion between times t_0 and t_1.

[2 marks]

$$S[q_i] = \int_{t_0}^{t_1} L \mathrm{d}t$$

A.2 Call the true trajectory of the particle $q_i(t)$. By considering variations away from this trajectory of the form $q_i(t) + \lambda\epsilon_i(t)$, formulate a mathematical statement of the principle of least action, explaining your reasoning.

[3 marks]

The principle of least action states that the classical trajectory of the particle extremises the action.
[1 mark]
Mathematically:

$$\left.\left(\frac{\partial S[q_i + \lambda\epsilon_i]}{\partial\lambda}\right)_{q_i, \epsilon_i}\right|_{\lambda=0} = 0$$

[1 mark].
The reason is that the principle states that the classical path extremises the action in the space of all trajectories, and an extremum by definition has a vanishing first derivative.
[1 mark]
N.B. the name is a bit of a misnomer based on the fact that 'extremises' often means 'minimises' in practice. Actually even 'extremise' is not fully correct either: the path followed is one along which the action is constant, so the variation is zero. We return to this later in the course.

A.3 Using the principle of least action, derive the Euler Lagrange equations describing the motion of the particle.

[5 marks]

$$S[q_i] = \int_{t_0}^{t_1} L(q_i, \dot{q}_i, t) \, \mathrm{d}t$$

so

$$S[q_i + \lambda\epsilon_i] = \int_{t_0}^{t_1} L(q_i + \lambda\epsilon_i, \dot{q}_i + \lambda\dot{\epsilon}_i, t) \, \mathrm{d}t$$

[1 mark].

$$\begin{aligned}
\frac{\partial S[q_i + \lambda\epsilon_i]}{\partial\lambda} &= \int_{t_0}^{t_1} \left\{ \frac{\partial L}{\partial q_i} \frac{\partial(q_i + \lambda\epsilon_i)}{\partial\lambda} + \frac{\partial L}{\partial\dot{q}_i} \frac{\partial(\dot{q}_i + \lambda\dot{\epsilon}_i)}{\partial\lambda} + \frac{\partial L}{\partial t}\frac{\partial t}{\partial\lambda} \right\} \mathrm{d}t \\
&= \int_{t_0}^{t_1} \left\{ \frac{\partial L}{\partial q_i}\epsilon_i + \frac{\partial L}{\partial\dot{q}_i}\dot{\epsilon}_i \right\} \mathrm{d}t
\end{aligned}$$

[1 mark].
Therefore

$$\left.\frac{\partial S\left[q_i+\lambda\epsilon_i\right]}{\partial\lambda}\right|_{\lambda=0}=0=\int_{t_0}^{t_1}\left\{\frac{\partial L}{\partial q_i}\epsilon_i+\frac{\partial L}{\partial\dot{q}_i}\dot{\epsilon}_i\right\}\mathrm{d}t.$$

[1 mark].
Integrate the second term by parts:

$$0=\int_{t_0}^{t_1}\frac{\partial L}{\partial q_i}\epsilon_i\mathrm{d}t-\int_{t_0}^{t_1}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}_i}\right)\epsilon_i\mathrm{d}t+\left[\frac{\partial L}{\partial\dot{q}_i}\epsilon_i\right]_{t_0}^{t_1}$$

but the boundary term is zero, by assumption: $\epsilon_i\left(t_0\right)=\epsilon_i\left(t_1\right)=0$.
[1 mark]
Therefore

$$0=\int_{t_0}^{t_1}\left\{\frac{\partial L}{\partial q_i}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}_i}\right)\right\}\epsilon_i\mathrm{d}t.$$

This is true for all $\epsilon_i\left(t\right)$. The only way that can be true is if the other part of the thing inside the integral is zero. Therefore

$$\frac{\partial L}{\partial q_i}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}_i}\right)=0.$$

[1 mark]

Section B: bringing together ideas from across the course.

Now consider a simple example of functional calculus in a different context.

The distance between two points is given by the functional

$$l\left[\boldsymbol{x}\right]=\int_{x_0}^{x_1}\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x \tag{6}$$

where

$$\boldsymbol{x}=\left(\begin{array}{c}x\\y\end{array}\right). \tag{7}$$

B.2 Explain the origin of Eq. 6.

[3 marks]

The distance between two points is a functional of the path taken. Specifically, it is the integral of the line elements along the path:

$$l\left[\boldsymbol{x}\right]=\int\mathrm{d}l$$

[1 mark].
Decomposing into cartesian co-ordinates, using Pythagoras' theorem we have

$$\mathrm{d}l^2=\mathrm{d}x^2+\mathrm{d}y^2$$

or

$$\mathrm{d}l=\sqrt{\mathrm{d}x^2+\mathrm{d}y^2}$$

(+ve since lengths are +ve). Therefore

$$l\left[\boldsymbol{x}\right] = \int \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}$$

[1 mark]
or, pulling the dx out,

$$l\left[\boldsymbol{x}\right] = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

[1 mark].

B.2 By considering paths $\boldsymbol{x}+\lambda\boldsymbol{\epsilon}$, find a differential equation describing the path minimising the distance between x_1 and x_2. Explain all the steps in your working.

[4 marks]

Hopefully the first part has made it clear that we seek something like the Euler Lagrange equation.

$$\begin{aligned} l\left[\boldsymbol{x}+\lambda\boldsymbol{\epsilon}\right] &= \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{\mathrm{d}\left(y+\lambda\epsilon_y\right)}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \\ &= \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x} + \lambda\frac{\mathrm{d}\epsilon_y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \end{aligned}$$

where ϵ_y is the y-component of the 2-component vector $\boldsymbol{\epsilon}$.
[1 mark]

$$\frac{\partial l\left[\boldsymbol{x}+\lambda\boldsymbol{\epsilon}\right]}{\partial\lambda} = \int_{x_0}^{x_1} \left\{\left(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x} + \lambda\frac{\mathrm{d}\epsilon_y}{\mathrm{d}x}\right)^2\right)^{-1/2} \left(\frac{\mathrm{d}y}{\mathrm{d}x} + \lambda\frac{\mathrm{d}\epsilon_y}{\mathrm{d}x}\right)\frac{\mathrm{d}\epsilon_y}{\mathrm{d}x}\mathrm{d}\right\} x$$

[1 mark]
and

$$\left.\frac{\partial l\left[\boldsymbol{x}+\lambda\boldsymbol{\epsilon}\right]}{\partial\lambda}\right|_{\lambda=0} = \int_{x_0}^{x_1} \left\{\left(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{-1/2} \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}\epsilon_y}{\mathrm{d}x}\right\}\mathrm{d}x = 0$$

[1 mark].
Integrating by parts,

$$0 = -\int_{x_0}^{x_1} \frac{\mathrm{d}}{\mathrm{d}x}\left\{\left(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{-1/2} \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\right\}\epsilon_y\mathrm{d}x + \left[\left(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{-1/2} \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\epsilon_y\right]_{x_0}^{x_1}$$

and the boundary term vanishes, as always, since we assume the variation vanishes at the end points. Therefore

$$0 = \int_{x_0}^{x_1} \frac{\mathrm{d}}{\mathrm{d}x}\left\{\left(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{-1/2} \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\right\}\epsilon_y\mathrm{d}x$$

and since this is true for all ϵ_y we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\left(1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{-1/2}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\right\}=0$$

[1 mark].

B.3 Solve your equation to find the shortest distance between the two points. Does this match your intuition?

[3 marks]

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\left(1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{-1/2}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\right\}=0$$

so

$$\left(1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{-1/2}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=C$$

with C constant.
[1 mark]
Therefore

$$\begin{aligned}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2&=C^2\left(1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)\\ \frac{\mathrm{d}y}{\mathrm{d}x}&=\frac{\pm C}{\sqrt{1-C^2}}\\ y&=C_1x+C_2\end{aligned}$$

with C_i constant.
[1 mark]
This is intuitive, as the shortest distance between two points is a straight line (in Euclidean space).
[1 mark]
NB for 1 mark not all that detail is needed in the first bit; any sensible statement that the solution is linear is fine.

Section C: more challenging.

C.1 The squared proper distance between two spacetime events in Minkowski space is

$$\mathrm{d}s^2=\mathrm{d}x^2-c^2\mathrm{d}t^2. \tag{8}$$

Find the path $t(x)$ that minimises the proper distance between two spacetime events.
[Hint: you may use your answers to Section B.]

[3 marks]

The hint, combined with our correct understanding in B.2, leads us to see that

$$l\left[x_\mu\right]=\int_{x_0}^{x_1}\sqrt{1-c^2\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)^2}\,\mathrm{d}x.$$

[1 mark].
This is exactly the same form as before, with

$$y \to ict.$$

[1 mark]
Therefore from B.3 we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\left(1 - c\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)^2\right)^{-1/2}\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)\right\} = 0$$

[1 mark]

C.2 What is the proper distance along this path?

[2 marks]

The solution to the equation is actually just as before: a straight line. Nothing has changed in the reasoning. And of course we know that the shortest proper distance between two points is obtained (only) by light:

$$x = ct$$

[1 mark].
We can therefore substitute into the 'action' to find

$$l\left[x_\mu\right] = \int_{x_0}^{x_1} \sqrt{1 - c^2\left(\frac{1}{c}\right)^2}\,\mathrm{d}x = 0.$$

[1 mark].
It's perhaps slightly counterintuitive, but that's the Minkowski metric for you!

2 Lagrangian Mechanics

Section A: mostly bookwork

Consider the action

$$S\left[q_i\right]=\int_{t_i}^{t_f} L\left(q_i,\dot{q}_i\right)\mathrm{d}t \tag{9}$$

where the Lagrangian L depends on time only implicitly via the generalised co-ordinates $q\left(t\right)$ and velocities $\dot{q}\left(t\right)$.

A.1 Consider a general variation of the action, δS. Using Eq. 9, and the chain rule, write an expression for δS. Make it clear what is held constant in each partial derivative.

[4 marks]

A.2 Hence, or otherwise, find an expression for the functional derivative

$$\frac{\delta S\left[\boldsymbol{q}\right]}{\delta q_i\left(t'\right)}.$$

Hint: You may use the fact that

$$\frac{\delta q_j\left(t\right)}{\delta q_i\left(t'\right)}=\delta^i_j\delta\left(t-t'\right) \tag{10}$$

where δ^i_j *is the Kronecker* δ*, and* $\delta\left(t-t'\right)$ *is a Dirac* δ *function.*

[4 marks]

A.3 Hence explain the origin of the Euler Lagrange equations

$$\frac{\partial L}{\partial q_i}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}=0. \tag{11}$$

[2 marks]

Section B: bringing together ideas from across the course

In relativistic problems space and time must be treated on equal footing. The action can be written in terms of a Lagrangian density $\mathcal{L}$:

$$S\left[\varphi\right]=\int \mathrm{d}t L=\int \mathrm{d}t\int \mathrm{d}^3\boldsymbol{x}\mathcal{L}\left(\varphi,\partial_\mu\varphi\right). \tag{12}$$

Here $\varphi\left(x^\mu\right)$ is a function of the spacetime co-ordinates

$$x^\mu=\begin{pmatrix} ct \\ \boldsymbol{x}\end{pmatrix}^\mu \tag{13}$$

and

$$\partial_\mu=\frac{\partial}{\partial x^\mu}=\begin{pmatrix} \frac{1}{c}\frac{\partial}{\partial t} \\ \frac{\partial}{\partial \boldsymbol{x}}\end{pmatrix}^\mu. \tag{14}$$

B.1 By considering the two variables of which the Lagrangian density is an explicit function, write an expression for the variation of the action δS.

[4 marks]

B.2 By setting

$$\frac{\delta S\left[\varphi\right]}{\delta\varphi\left(x^{\nu}\right)}=0 \tag{15}$$

derive the relativistic Euler Lagrange equations.
Hint: you may use the fact that

$$\frac{\delta\varphi\left(x^{\mu}\right)}{\delta\varphi\left(x^{\nu}\right)}=\delta\left(x^{\mu}-x^{\nu}\right). \tag{16}$$

[4 marks]

B.3 Consider the action

$$S\left[\varphi\right]=\int\mathrm{d}t\int\mathrm{d}^{3}\boldsymbol{x}\left\{\frac{1}{2}\partial_{\mu}\varphi\partial^{\mu}\varphi-\frac{m^{2}c^{2}}{2\hbar^{2}}\varphi^{2}\right\} \tag{17}$$

where a sum over repeated indices μ is assumed. Show that the Euler Lagrange equation for this action is the Klein Gordon equation

$$\left(\frac{1}{c^{2}}\frac{\partial^{2}}{\partial t^{2}}-\nabla^{2}+\frac{m^{2}c^{2}}{\hbar^{2}}\right)\varphi=0. \tag{18}$$

Hint: you may use the fact that

$$\partial_{\mu}\partial^{\mu}=\frac{1}{c^{2}}\frac{\partial^{2}}{\partial t^{2}}-\nabla^{2}. \tag{19}$$

[2 marks]

Section C: more challenging

C.1 By writing the function φ as

$$\varphi\left(ct,\boldsymbol{x}\right)=\phi\left(ct,\boldsymbol{x}\right)\exp\left(-imc^{2}t/\hbar\right) \tag{20}$$

where mc^2 is the rest mass of the particle, show that the Klein Gordon equation reduces to the Schroedinger equation in the non-relativistic limit. State any assumptions that you make.

[5 marks]

Answers to Q2

Section A: mostly bookwork

Consider the action

$$S\left[q_i\right] = \int_{t_i}^{t_f} L\left(q_i, \dot{q}_i\right) \mathrm{d}t \tag{21}$$

where the Lagrangian L depends on time only implicitly via the generalised co-ordinates $q\left(t\right)$ and velocities $\dot{q}\left(t\right)$.

A.1 Consider a general variation of the action, δS. Using Eq. 21, and the chain rule, write an expression for δS. Make it clear what is held constant in each partial derivative.

[4 marks]

$$\begin{aligned}\delta S &= \int_{t_i}^{t_f} \delta L\left(q_i, \dot{q}_i, t\right) \mathrm{d}t \\ &= \int_{t_i}^{t_f} \left\{ \left(\frac{\partial L}{\partial q_i}\right)_{\dot{q},t} \delta q_i + \left(\frac{\partial L}{\partial \dot{q}_i}\right)_{q,t} \delta \dot{q}_i \right\} \mathrm{d}t\end{aligned}$$

[1 mark] each for the two terms
[1 mark] each for the two pairs held constant. Note: holding t constant is unnecessary as there is no explicit time dependence, so the mark is received whether it is there or not.

A.2 Hence, or otherwise, find an expression for the functional derivative

$$\frac{\delta S\left[\boldsymbol{q}\right]}{\delta q_i\left(t'\right)}.$$

Hint: You may use the fact that

$$\frac{\delta q_j\left(t\right)}{\delta q_i\left(t'\right)} = \delta^i_j \delta\left(t - t'\right) \tag{22}$$

where δ^i_j is the Kronecker δ, and $\delta\left(t-t'\right)$ is a Dirac δ function.

[4 marks]

$$\begin{aligned}\delta S &= \int_{t_i}^{t_f} \left\{ \left(\frac{\partial L}{\partial q_i}\right)_{\dot{q},t} \delta q_i + \left(\frac{\partial L}{\partial \dot{q}_i}\right)_{q,t} \delta \dot{q}_i \right\} \mathrm{d}t \\ \frac{\delta S}{\delta q_i\left(t'\right)} &= \int_{t_i}^{t_f} \left\{ \left(\frac{\partial L}{\partial q_i}\right)_{\dot{q},t} \frac{\delta q_i\left(t\right)}{\delta q_i\left(t'\right)} + \left(\frac{\partial L}{\partial \dot{q}}\right)_{q,t} \frac{\delta \dot{q}\left(t\right)}{\delta q_i\left(t'\right)} \right\} \mathrm{d}t\end{aligned}$$

[1 mark]
2nd term by parts:

$$\frac{\delta S}{\delta q_j\left(t'\right)} = \int_{t_i}^{t_f} \left\{ \left(\frac{\partial L}{\partial q_i}\right)_{\dot{q},t} \frac{\delta q_i\left(t\right)}{\delta q_j\left(t'\right)} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right)_{q,t} \frac{\delta q_i\left(t\right)}{\delta q_j\left(t'\right)} \right\} \mathrm{d}t + \left[\left(\frac{\partial L}{\partial \dot{q}_i}\right)_{q,t} \frac{\delta q_i\left(t\right)}{\delta q_j\left(t'\right)}\right]_{t_i}^{t_f}$$

[1 mark]
and the boundary term is assumed zero because of the boundary conditions.
Then we use the stated expression:

$$\frac{\delta S}{\delta q_j\left(t'\right)} = \int_{t_i}^{t_f} \left\{ \left(\frac{\partial L}{\partial q_i}\right)_{\dot{q},t} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right)_{q,t} \right\} \delta_i^j \delta(t-t')\mathrm{d}t$$

[1 mark]
Giving the result

$$\frac{\delta S}{\delta q_j\left(t'\right)} = \left(\frac{\partial L}{\partial q_j}\right)_{\dot{q}} - \frac{\mathrm{d}}{\mathrm{d}t'}\left(\frac{\partial L}{\partial \dot{q}_j}\right)_{q}$$

[1 mark]

A.3 Hence explain the origin of the Euler Lagrange equations

$$\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = 0. \tag{23}$$

[2 marks]

The principle of least action says that classical paths extremise the action.
[1 mark]
Hence the l.h.s. is 0, and the equations follow.
[1 mark]

Section B: bringing together ideas from across the course

In relativistic problems space and time must be treated on equal footing. The action can be written in terms of a Lagrangian density $\mathcal{L}$:

$$S\left[\varphi\right] = \int \mathrm{d}t L = \int \mathrm{d}t \int \mathrm{d}^3\boldsymbol{x}\mathcal{L}\left(\varphi, \partial_\mu\varphi\right). \tag{24}$$

Here $\varphi\left(x^\mu\right)$ is a function of the spacetime co-ordinates

$$x^\mu = \begin{pmatrix} ct \\ \boldsymbol{x} \end{pmatrix}^\mu \tag{25}$$

and

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \begin{pmatrix} \frac{1}{c}\frac{\partial}{\partial t} \\ \frac{\partial}{\partial \boldsymbol{x}} \end{pmatrix}^\mu. \tag{26}$$

B.1 By considering the two variables of which the Lagrangian density is an explicit function, write an expression for the variation of the action δS. Make it clear what is held constant in each partial derivative.

[4 marks]

$$
\begin{aligned}
\delta S &= \int \mathrm{d}t \int \mathrm{d}^3\boldsymbol{x} \delta \mathcal{L}\left(\varphi, \partial_\mu \varphi\right) \\
&= \int \mathrm{d}t \int \mathrm{d}^3\boldsymbol{x} \left\{ \left(\frac{\partial \mathcal{L}}{\partial \varphi}\right)_{\partial_\mu \varphi} \delta\varphi + \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi}\right)_{\varphi} \delta \partial_\mu \varphi \right\}.
\end{aligned}
$$

As before, **[1 mark]** for each term and **[1 mark]** for each correct statement as to what is held constant.

B.2 By setting

$$
\frac{\delta S\left[\varphi\right]}{\delta \varphi\left(x^\nu\right)} = 0 \tag{27}
$$

derive the relativistic Euler Lagrange eqautions.
Hint: you may use the fact that

$$
\frac{\delta \varphi\left(x^\mu\right)}{\delta \varphi\left(x^\nu\right)} = \delta\left(x^\mu - x^\nu\right). \tag{28}
$$

[4 marks]

$$
\frac{\delta S}{\delta \varphi\left(x^\nu\right)} = \int \mathrm{d}t \int \mathrm{d}^3\boldsymbol{x} \left\{ \left(\frac{\partial \mathcal{L}}{\partial \varphi}\right)_{\partial_\mu \varphi} \frac{\delta \varphi\left(x^\mu\right)}{\delta \varphi\left(x^\nu\right)} + \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi}\right)_{\varphi} \frac{\delta \partial_\mu \varphi\left(x^\mu\right)}{\delta \varphi\left(x^\nu\right)} \right\}
$$

[1 mark]

$$
0 = \int \mathrm{d}t \int \mathrm{d}^3\boldsymbol{x} \left\{ \left(\frac{\partial \mathcal{L}}{\partial \varphi}\right)_{\partial_\mu \varphi} \frac{\delta \varphi\left(x^\mu\right)}{\delta \varphi\left(x^\nu\right)} + \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi}\right)_{\varphi} \frac{\delta \partial_\mu \varphi\left(x^\mu\right)}{\delta \varphi\left(x^\nu\right)} \right\}
$$

[1 mark]. Integrate the second term by parts:

$$
0 = \int \mathrm{d}t \int \mathrm{d}^3\boldsymbol{x} \left\{ \left(\frac{\partial \mathcal{L}}{\partial \varphi}\right)_{\partial_\mu \varphi} \frac{\delta \varphi\left(x^\mu\right)}{\delta \varphi\left(x^\nu\right)} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi}\right)_{\varphi} \frac{\delta \varphi\left(x^\mu\right)}{\delta \varphi\left(x^\nu\right)} \right\} + \left[\boldsymbol{\left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi}\right)_{\varphi} \frac{\delta \varphi\left(x^\mu\right)}{\delta \varphi\left(x^\nu\right)}} \right]_{\boldsymbol{x}_i^\mu}^{\boldsymbol{x}_f^\mu} \tag{29}
$$

and the boundary term disappears by assumption.
[1 mark]
Use the stated relation to find

$$
\begin{aligned}
0 &= \int \mathrm{d}t \int \mathrm{d}^3\boldsymbol{x} \left\{ \left(\frac{\partial \mathcal{L}}{\partial \varphi}\right)_{\partial_\mu \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi}\right)_{\varphi} \right\} \delta\left(x^\mu - x^\nu\right) \\
0 &= \left(\frac{\partial \mathcal{L}}{\partial \varphi}\right)_{\partial_\mu \varphi} - \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial \partial_\nu \varphi}\right)_{\varphi}
\end{aligned}
$$

[1 mark].

B.3 Consider the action

$$
S\left[\varphi\right] = \int \mathrm{d}t \int \mathrm{d}^3\boldsymbol{x} \left\{ \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{m^2 c^2}{2\hbar^2} \varphi^2 \right\} \tag{30}
$$

where a sum over repeated indiced μ is assumed. Show that the Euler Lagrange equation for this action is the Klein Gordon equation

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\varphi = 0. \tag{31}$$

Hint: you may use the fact that

$$\partial_\mu \partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2. \tag{32}$$

[2 marks]

$$0 = \left(\frac{\partial \mathcal{L}}{\partial \varphi}\right)_{\partial_\mu \varphi} - \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial \partial_\nu \varphi}\right)_{\varphi}$$
$$0 = -\frac{m^2c^2}{\hbar^2}\varphi - \partial_\nu \partial^\nu \varphi$$

[1 mark] and so

$$\left(\partial_\nu \partial^\nu + \frac{m^2c^2}{\hbar^2}\right)\varphi = 0$$
$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\varphi = 0$$

[1 mark].

Section C: more challenging

C.1 By writing the function φ as

$$\varphi\left(ct, \boldsymbol{x}\right) = \phi\left(ct, \boldsymbol{x}\right)\exp\left(-imc^2 t/\hbar\right) \tag{33}$$

where mc^2 is the rest mass of the particle, show that the Klein Gordon equation reduces to the Schroedinger equation in the non-relativistic limit. State any assumptions that you make.

[5 marks]

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\varphi = 0 \tag{34}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\phi \exp\left(-imc^2 t/\hbar\right) = 0 \tag{35}$$

[1 mark]. Focus on the time derivative:

$$\frac{\partial^2}{\partial t^2}\left(\phi \exp\left(-imc^2t/\hbar\right)\right) = \frac{\partial}{\partial t}\left(\dot{\phi}\exp\left(-imc^2t/\hbar\right) - \frac{imc^2}{\hbar}\phi\exp\left(-imc^2t/\hbar\right)\right) \tag{36}$$

$$= \left\{\ddot{\phi} - 2\frac{imc^2}{\hbar}\dot{\phi} + \left(\frac{imc^2}{\hbar}\right)^2\phi\right\}\exp\left(-imc^2t/\hbar\right). \tag{37}$$

[1 mark]
In the non-relativistic limit, mc^2 is the dominant term, so drop the $\ddot{\phi}$ term.
[1 mark]
This gives

$$\left(-2\frac{im}{\hbar}\frac{\partial}{\partial t} - \frac{m^2c^2}{\hbar^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\phi\exp\left(-imc^2t/\hbar\right) = 0$$

[1 mark]. Rewriting,

$$i\hbar\frac{\partial}{\partial t}\varphi = -\frac{\hbar^2}{2m}\nabla^2\varphi$$

as required
[1 mark].

3 Lagrangian Mechanics (assorted questions not in exam style)

3.1 Connecting classical to quantum mechanics

The Hamiltonian in classical mechanics can be found from the Lagrangian using a Legendre transform:

$$H = p_i \dot{q}^i - L \tag{38}$$

where

$$p_i \triangleq \frac{\partial L}{\partial \dot{q}^i}. \tag{39}$$

Hamilton's equations of motion are then

$$\dot{q} = \frac{\partial H}{\partial p}; \qquad \dot{p} = -\frac{\partial H}{\partial q}. \tag{40}$$

Finally, the Poisson bracket is defined as

$$\{f, g\} \triangleq \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p^i} - \frac{\partial f}{\partial p^i}\frac{\partial g}{\partial q_i}. \tag{41}$$

3.11 What is held constant in each of the seven partial derivatives?

[7 marks]

3.12 Prove the relations given in the lectures:

$$\dot{q}_i = \{q_i, H\} \tag{42}$$
$$\dot{p}_i = \{p_i, H\}. \tag{43}$$

[4 marks]

3.13 Assume we have a function with the dependence $f(q_i, p_i, t)$. Derive Hamilton's equation of motion:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{f, H\} + \left(\frac{\partial f}{\partial t}\right)_{q_i, p_i}. \tag{44}$$

[4 marks]

3.14 Explain the difference between the Heisenberg and Schroedinger pictures of quantum mechanics. You may need to remind yourself of the third year quantum course.

[4 marks]

3.15 Using the labels H and S to label the two pictures, we have the following relationship between operators:

$$\hat{A}_H(t) = e^{-i\hat{H}t/\hbar}\hat{A}_S(t)\, e^{i\hat{H}t/\hbar} \tag{45}$$

where the operator in the Schroedinger picture has an explicit time dependence (which is rarely a case we consider). Assuming the Hamiltonian is time independent, derive the Heisenberg equation of motion:

$$\frac{\mathrm{d}\hat{A}_H}{\mathrm{d}t} = \frac{1}{i\hbar}\left[\hat{A}_H, \hat{H}\right] + \left(\frac{\partial \hat{A}_S}{\partial t}\right)_H$$

explaining the meaning of the final term.

[4 marks]

3.17 Using the Heisenberg equation of motion, prove Ehrenfest's theorem:

$$\frac{\mathrm{d}\left\langle \hat{A} \right\rangle}{\mathrm{d}t} = \frac{1}{i\hbar}\left\langle \left[\hat{A}, \hat{H}\right] \right\rangle + \left\langle \frac{\partial \hat{A}}{\partial t} \right\rangle \tag{46}$$

where

$$\left\langle \hat{A} \right\rangle \triangleq \langle \psi | \hat{A} | \psi \rangle \tag{47}$$

for an arbitrary state $|\psi\rangle$.

[2 marks]

3.18 Why were we able to drop the H subscript in 3.17?

[1 mark]

3.19 Find a friend and discuss the relationship between classical constants of motion, and good quantum numbers.